

Realism-Completeness-Universality interpretation of quantum mechanics

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Abstract: The paper reviews and discusses four ideas scattered in previous papers of the author. First, objective properties of quantum systems are not associated with observables but are defined by preparations. Second, measurable results of classical theories are not sharp. All such results can be obtained as high-entropy limits of quantum mechanics. Third, a careful study of exchange symmetry requires a modification of the theory of measurement. Fourth, screens and detectors are closely associated with state reduction. A new, rather improved understanding of quantum mechanics is achieved without any deep changes of the theory itself. In the framework of a model approach to the philosophy of science, the RCU interpretation is developed step by step by adding specific supplementary hypotheses to the Minimum Interpretation of quantum mechanics.

Introduction

The non-relativistic quantum mechanics was borne during 1920's, was described in the books by Heisenberg, Dirac, Pauli and von Neumann (later editions of these books are [37, 12, 56, 55]) and has been in use unchanged in its main concepts and methods until now. It is a very successful theory and the source of our present knowledge on the structure of matter and its properties. The practical consequences of the knowledge have completely changed the daily life of an average human.

Moreover, the theoretical structure of quantum mechanics is marvellous and beautiful. The basic concepts and the associated mathematical entities can be briefly listed as follows.

1. Each quantum phenomenon or experiment can be split into the so-called preparation and the so-called registration. Preparations and registrations are described by classical models.
2. Each quantum systems is associated with a copy of Hilbert space that carries a unitary representation of a central extension of Galilean group.
3. A state of the system is described by a positive operator with trace 1 on the system Hilbert space.
4. An observable of the system is described by a self-adjoint operator on the system Hilbert space.
5. Values of an observable obtained by a registration are eigenvalues of the observable. The probability of registering a given eigenvalue of an observable on a prepared state is calculated from the state and the observable by the so-called Born rule.
6. The most important observables of the system are the generators of the space-time symmetry group.
7. The Hilbert space of the system composed of two system of different type is the tensor product of the Hilbert spaces of the subsystems.
8. Subsystems of the same type cannot be distinguished from each other. This leads to the symmetry under the permutation group of the subsystems, the so-called exchange symmetry.

In addition, quantum mechanics is a highly symmetric theory. One of its symmetry groups is the infinitely dimensional transformation group of (generalized) Hilbert-space bases. This is comparable to the symmetry of the classical mechanics

with respect to canonical transformations. The second symmetry group is formed by the space-time transformations. Finally, the third one is the exchange-symmetry group, unknown in classical theories.

Let us turn to interpretation of this apparatus. There is a popular belief among physicists that quantum states refer to individual quantum systems (Realism), that quantum mechanics gives a complete description of the micro-world (Completeness) and that it is applicable to all physical objects (Universality). This agrees more or less with the so-called Dirac-von-Neumann version of the Copenhagen interpretation. Our aim is to reformulate this interpretation more precisely by explicitly stating what assumptions it adds to the so-called Minimum Interpretation, which could be roughly identified with Bohr's version of the Copenhagen interpretation in the rigorous form given to it by [51]. Minimum Interpretation is also explained and applied in textbooks [58] or [1].

We are going to study what can be added to the Minimum Interpretation and concentrate all such supplementary assumptions into several specific hypotheses, called Trial Hypotheses (TH). Some of such TH's are well known, but we will also introduce new and sometimes rather heretic ideas. In each case, we shall study the consequences of such changes, localize new problems and try to remove them. The resulting understanding of quantum mechanics will be called Realism-Completeness-Universality (RCU) interpretation.

The first bunch of TH's concerns the realism, Chapter 1. We shall first assume that properties defined uniquely by preparation are objective properties of the prepared system. This is a strong enhancement of the notion that states refer to individual quantum systems. Second, the values of observables are not objective but only created by registrations. This removes the contradictions that result from the assumption that values of observables exist before, and are only revealed by, registrations. Moreover, we are going to embed the realist notion of quantum mechanics into a specific branch of philosophy of science: the so-called Constructive Realism [20], which views physical theories as classes of models, each of which ought to describe some aspect of reality in an approximate way. We shall find that the language of models allows to formulate many difficult ideas in a precise and clear way.

As for the completeness, we shall assume that there are no unknown causes beyond the probabilities given by the Born rule. At least, quantum mechanics does not identify such causes. This does not lead to any internal logical problem, it just contradicts the philosophy of determinism. We call this assumption *Completeness Hypothesis*.

If we want quantum mechanics to be universally applicable, then it has to explain the so-called classical properties. Newtonian mechanics, Maxwellian electrody-

ics, phenomenological thermodynamics and classical chemistry are classical theories. They construct models of real objects and these models describe classical properties, such as position and momentum, electric and magnetic fields or volume, pressure and temperature as well as chemical composition. Certain sets of such properties define (classical) *states* and certain properties are values of quantities that can be called (classical) *observables*.

The classical theories are not obsolete or made invalid by quantum mechanics, but they describe certain aspects of certain physical objects in certain approximation. Indeed, they are still in use and successful, if their accuracy is sufficient for given aims, because the mathematics of the classical theories is usually much simpler than the corresponding quantum calculation would be. However, we have then to give a quantum-mechanical explanation of the successful classical predictions.

We are going to give an explanation that is rather different from what is usually assumed. We shall first postulate that a (real) physical object can have both classical and quantum models. Second, that the states of the classical model are associated with some high-entropy quantum states and third, that classical properties are then some properties of these high-entropy states. This must also hold for Newtonian mechanics and can be achieved by requiring the mechanics to describe only measurable aspects of “mechanical” reality. To this aim, the mathematical theory of the so-called maximum-entropy (ME) packets is introduced, Chapters 2 and 3. Chapter 2 shows that Newtonian mechanics is then analogous to statistical thermodynamics. The main result of Chapter 3 is that the classical limit is a well-defined high-entropy limit.

Finally, a well-known paradox results from the assumption that states refer to individual systems, namely the existence of two different kinds of dynamics: the unitary evolution generated by the generator of the time translation of Galilean group, and the creation of definite values of observables by a registration process, the so-called state reduction. Thus, we have to deal with the problem of the two dynamics. This will be done in Chapters 4 and 5. Chapter 4 shows that current quantum theory of measurement in all its variants is deficient because it neglects the strong influence of the exchange symmetry on measurement processes that necessarily follows from a consequent application of quantum mechanical principles. A consequence of this influence is that registration apparatuses can work only if they are *incomplete*. That is, Born rule does not hold for registered values on all states. Moreover, any preparation must elevate the prepared system from the sea of identical particles. We say that such a system has a *separation status*. The chapter gives a rigorous definition of the separation status and introduces the mathematics needed to describe evolution of such systems.

Chapter 5 gives a reformulation of the quantum theory of measurement so that

it becomes compatible with Chapters 3 and 4. It analyzes the registration processes and postulates that each registration apparatus contains a special object called *detector*. It then shows that the state reduction occurs inside the detector or inside the screen if they cause a loss of separation status. Thus, the existence of the two dynamics is not removed but objective conditions of when the state reduction occurs and where it happens, are specified. This leads to observable phenomena, at least in principle. Finally, Chapter 6 contains a review of all Trial Hypotheses and a concluding discussion.

Most of these ideas have been published some time ago ([25]–[36]). The present paper is written more carefully than these papers, corrects some errors and fills some gaps. Moreover, the new exposition is appreciably simplified and the focus is on examples and models rather than on attempts to formulate the most general statements. This is in agreement with the adopted kind of philosophy of science. Thus, the exposition is better accessible and I believe that the paper can be read by students that have finished an ordinary course of quantum mechanics. In any case, there are excellent textbooks such as [58] and [1] that can be recommended as preparatory reading.

The present paper not only collects some ideas that are scattered in literature but also answers the question whether the whole quantum mechanics can be formulated in a coherent way so that it is compatible with these ideas. The ultimate aim is to show that our understanding of quantum mechanics can be rather improved without any deep changes of quantum mechanics itself.

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Chapter 1

Objective properties

1.1 Realism in philosophy of science

A realist interpretation of a physical theory is a subtler and deeper problem than an answer to the question of whether the world exists for itself rather than being just a construction of our mind. This question can always be answered in positive without any danger of falsification.

There is a more interesting question. Every physical theory introduces some general, abstract concepts. For example, Newtonian mechanics works with mass points, their coordinates, momenta and their dynamical trajectories. The question is, whether such concepts possess any counterparts in the real world. On the one hand, it seems very plausible today that mass points and their sharp trajectories cannot exist and are at most some idealisations. On the other hand, if we are going to understand a real object, such as a snooker ball moving on a table, then we can work with a construction that uses these concepts but is more closely related to the reality. For example, we choose a system of infinitely many mass points forming an elastic body of a spherical shape and calculate the motion of this composite system using Newton's laws valid for its constituent points. Then, some calculated properties of such a model can be compared with interesting observable properties of the real object. Thus, even if the general concepts of the theory do not describe directly anything existing, a suitable model constructed with the help of the general concepts can account for some aspects of a real object.

Motivated by this observation, one can divide any physical theory into two parts. First, there is a treasure of successful *models*. Each model gives an *approximative* representation of some *aspects* of a real object [20] (Giere uses words “degree” and “respects” instead). Historically, models form a primary but open part of the theory. For example, in Newtonian mechanics, the Sun and its planets were carefully observed by Tycho de Brahe and then some model of it was constructed by Kepler.

Apparently, Newton was able to calculate accelerations and doing so for Kepler trajectories, he might discover that they pointed towards the Sun. Perhaps this led to the Second Law. The hydrogen atom had a similar role in quantum mechanics.

Second, there is a general *language* part. If we restrict ourselves to physics, it contains the mathematical structure of state space, conditions on trajectories in the state space, their symmetries and the form of observables [15]. It is obtained by generalisation from the study of models but it is also an instrument of further model construction and a tool for unification of the models. For example, in Newtonian mechanics, the phase space is the state space, Newton's dynamical equations are the conditions on trajectories, Galilean transformations are symmetries and real functions on the phase space, such as a Hamiltonian, are observables.

A model is constructed as a particular subset of trajectories in a particular state space as well as a choice of important observables. For example, to describe the solar system, assumptions on the number of bodies, their point-like form, their masses, the form of gravitational force and certain class of their trajectories can be made if we want to construct a model. The observed positions of the planets ought then to match the theoretical trajectories of the model within certain accuracy. Thus, each model is associated with a real object. The model always contains simplifying assumptions, always holds only for some aspects of the object and only within some approximation. The accuracy of the approximation that is referred to may be unknown and is different from the accuracy of measurements that can be performed at a given historical stage. This is measurable and can be expressed numerically by statistical variances.

The models of a given theory are not predetermined by the language part but obtained in the historical evolution and dependent on observation of real objects. Indeed, on the one hand, the language part can also be used to construct models that do not have any real counterparts. On the other, the model part is steadily evolving and never closed. For example, a satisfactory quantum model of high temperature superconductivity is not yet known. This is why the treasure of successful models is an independent and, in fact, the fundamental part of any theory.

Such an approach lies somewhat within the recent trend of the philosophy of science that defines a theory as a class of models (see, e.g., [70, 68, 15, 20, 10]). It can be said that it combines ideas of Constructive Realism by Ronald Giere with van Fraassen notions of state space and symmetries [15] as a basis of the general language part. It is important that Constructive Realism is immune to the usual objections against naive realism. Naive realism is roughly characterised by the statement: "The world is as it is perceived", which is obviously wrong.

To proceed any further, we have to clarify the relation between real objects and their theoretical models. What is a real object?

An object (we shall leave out the adjective “real”) is assumed to be the cause of certain empirical experience we can have. These observations also motivate assumptions about certain objective properties of the object. Objective means here that the properties can be ascribed to the object alone, each such ascription being a kind of *ontological hypothesis*. Lead by such heuristic ideas, we shall later define objective properties of quantum objects in a more precise way simply by listing the mathematical entities that can describe them.

A few words have perhaps to be said on ontological hypotheses. As is well-known, the objective existence of anything cannot be proved (even that of the chair on which I am now sitting, see, e.g., Ref. [13], where this old philosophical tenet is explained from the point of view of a physicist). Thus, all such statements are only hypotheses. However, a sufficiently specific ontological hypothesis may lead to contradictions with some observations and some hypotheses which do not lead to contradictions may be useful. For example, the objective existence of the chair nicely explains why we all agree on its properties. Similarly, the assumption that quantum systems possess certain objective properties might be useful for the quantum theory of classical properties or for a solution of the problem of quantum measurement.

We shall say “Object a has a property A ” if A is an objective property of a . An object must then satisfy the following requirements:

- O1** An object has enough objective properties: all objective properties of an object define the object uniquely in the sense that we can recognise the same object at different times and different space position, as well as distinguish it from other objects in the environment.
- O2** The proposition “Object a has a property A ” for a given object a and for all its objective properties A is always either true or false.

As an example, consider that dice I am holding in my hand. The dice is defined by its geometry, chemical composition, and colours. It can exist during several years and take part in different processes at different positions. Its positions at different times are objective properties that do not define it. If I toss the dice, some number will fall. Such number is an example of a property that is not an objective property of the dice but it is an objective property of the toss.

Theoretical models of a given object ought to explain the observed properties of the object. We stress again that the models do not describe the assumed object exactly and completely. Such an exact and complete knowledge of any given object will perhaps never be achieved. On the one hand, for a given object, there can be several models that can differ in sophistication and accuracy. On the other hand, one model can describe a whole class of objects.

1.2 Application to quantum mechanics

However, the approach is not as easily applied to quantum mechanics as it is to Newtonian mechanics. A question looms large at the very start: What are the object, of which quantum mechanics makes models? Could we leave the Minimum Interpretation and assume that such objects are not just classical apparatuses but also some microscopic objects met empirically in preparations and registrations? The character of such assumptions has been analysed by Giere [20], p. 115. Giere distinguishes trial suppositions about reality, which are still the subject of investigations and experiments, from assured knowledge about real objects, which can be recognized by ability to control and manipulate the object and thus to use it in investigations and experiments on other problems.

Our notion of properties allows us to express a similar idea. An existing individual object must have a sufficient number of objective properties so that they satisfy requirement O1. Then, we may know enough about the object in order to be able to manipulate and control it. Another idea of a similar kind has been introduced by Gisin [21]:

A theory is *realistic* if and only if, according to the mathematical structure of the theory, the collection of all physical quantities written in the system unambiguously determines the probabilities of all possible measurement outcomes.

Here, “physical quantities written in the system” means what we termed “objective properties”.

In Newtonian mechanics, any mechanical object is modelled by a mechanical system. The values of all point masses and their coordinates and momenta of the system distinguish different objects. On the one hand, Newtonian coordinates and momenta define a state, that is a point in the phase space of the system, on the other, they are observables. In Newtonian mechanics, both states and values of observables are directly associated with, or explain, objective properties of a physical object without any danger of contradictions.

The situation in quantum mechanics is more complicated. Let us first formulate the ontological hypotheses that are compatible with the Minimum Interpretation.

1. Specific processes running in an arrangement of classical apparatuses are quantum measurements. A subset of the apparatuses are preparation, another are registration ones. The apparatuses are objects, their classical properties distinguishing them from any other such objects.
2. A system type and a prepared state can be considered as certain classes of

preparation apparatuses. An observable can be considered as a class of registration apparatuses.

3. An *individual* measurement process is defined as such that results in a unique definitive registration outcome: a unique value of a quantum observable. Quantum mechanics is a set of rules allowing the computation of probabilities for outcomes of individual registrations that follow specified preparations [58], p. 13.

Actually, there can be different states associated with an individual measurement process, each referring to a different time instant of the process.

Now, as promised in the Introduction, we are going to augment the Minimum Interpretation by further trial hypotheses.

Definition 1.1 *A structural property of a quantum system is a property that is common to all systems of the same type.*

For example, mass, spin and electric charge are structural properties of particles while composition and Hamiltonian are those of composite systems.

The basis of the realist part of RCU interpretation is:

Trial Hypothesis 1.1 *A quantum object is defined by a preparations. The objects are thus distinguished from each other by the properties that are determined by their preparations. These include the structural properties describing a system type, the prepared state and properties that are uniquely defined by the state. Objects of non-relativistic quantum mechanics can be classified into electrons, neutrons, nuclei, atoms, molecules and their composites.*

What we have added is that system types and prepared states are objective properties of microscopic objects. Such objects can take part in measuring processes but are then different from arrangements of classical apparatuses and measurement processes. Further properties, different from structural ones and states, will be added in Section 2.3 by TH 1.5.

Sometimes, TH 1.1 meets one of the following two questions. First, how can the Hypothesis be applied to cosmology, when there was nobody there at the Big Bang to perform any state preparation? Second, a state preparation is an action of some human subject; how can its result be an objective property? Both objections originate in a too narrow view of preparation: it is not necessarily a human activity. Moreover, if we don't know the preparation, we can still assume that the considered object is in some state, for example in the case of cosmology. This is, in principle, a testable hypothesis. Actually, the second objection is not much more than a pun. It is not logically impossible that a human manipulation of a object results in

an objective property of the object. For example, pushing a snooker ball imparts it a certain momentum and angular momentum that can then be assumed to be objective properties of the pushed ball.

We distinguish quantum objects and quantum systems. Quantum object is an object of which quantum model is constructed. Quantum system is a part of the quantum model. A system is distinguished from other systems by some symbol and is mathematically described by a Hilbert space that carries a representation of a spacetime group. We shall denote objects by calligraphic capital letters such as \mathcal{S} and systems by capital letters such as S . Because of the exchange symmetry, a quantum system is just an auxiliary mathematical notion that has no really existing counterpart. This will be clarified in Chapter 4. Moreover, there are also fully abstract quantum systems, such as the centre of mass of an isolated composite system.

As a comment to requirement O2 applied to microscopic objects of quantum mechanics (this idea is due to Günter Ludwig [51]), let us clarify the relation to the quantum logic [3], which does not satisfy requirement O2. The properties studied by the quantum logic are values of some observables. But values of observables are analogous to values obtained by tossing a dice: each such value is not a property of the dice alone but also of a particular toss. They cannot be attributed to the object alone. The fact that the mathematics of the quantum logic is still a beautiful kind of algebra is due to von Neumann's smart choice of very special observables (projections) that satisfy specific geometric relations. Thus, on the one hand, they are not properties of one object, on the other hand, they are not properties of all possible registrations on a given state.

According to TH 1.1, a sufficient condition for a property of a quantum system to be objective is that its value can be uniquely determined by a preparation according to the rules of standard quantum mechanics. The “value” is the value of the mathematical expression that describes the property and it may be more general than just a real number. To observe an objective property, many registrations of different kinds may be necessary.

We consider properties that are complex in the following sense [36]:

1. Their values may be arbitrary mathematical entities (sets, maps between sets, etc.). For example, the Hamiltonian of a closed quantum system involves a relation between energy and some other quantities of the system. This relation is an example of such a complex property.
2. Their values need not be directly obtained by individual registrations. For example, to measure a cross-section a whole series of scattering experiments must be done. Thus, their values need not possess probability distribution but may be equivalent to, or derivable from, probability distributions.

Such complex properties are nothing new. For example, in Newtonian mechanics, a value of any given observable O for any mechanical object is never “known exactly” but only as an expectation value with a variance (i.e., mean quadratic deviation), and such a value is only obtained by many registrations. The statement that there *is* an exact value and the expectation value with the variance is only due to the inaccuracy of measuring techniques is, in fact, one of the simplifying hypotheses of the language part of the theory that need not have anything to do with reality. This will be utilised in Chapter 2.

A value of an observable is an outcome of an individual registration performed on an already existing quantum object. In this way, RCU interpretation leads to a separation of the prepared object from its registration. This motivates postulating more for registrations than is directly observed (such directly observed properties are individual randomness and large-number regularity of registered values) and than would be required by a strict adherence to the Minimum Interpretation. Thus, we arrive at our second Trial Hypothesis.

Trial Hypothesis 1.2 (*Outcomes Created by Registration*) *The outcome of an individual registration performed on a quantum object \mathcal{S} in state T is in general only created during the registration. It is an objective property of the whole registration process.*

The opposite conjecture, that every possible outcome is always already determined before any registration, means that each quantum object has some further objective properties that are not uniquely determined by its preparations (the so-called “hidden variables”). As quantum mechanics does not determine these further properties, the conjecture about their existence contradicts the Completeness Hypothesis (see the Introduction). Moreover, this conjecture is also directly incompatible with quantum mechanics and a number of fine experiments concerning Bell inequalities [58, 1] and contextuality [58, 1, 8]. Of course, in some exceptional cases, a value of an observable is determined before its registration (if the state is an eigenstate of the observable).

To see the real meaning of TH 1.2, we just apply it to values of position. Position of a given quantum system is an observable and its eigenvalues can be measured. However, if a position value \vec{x} is an outcome of a registration on a state, this does not imply that some real part of the registered object has been at the point \vec{x} immediately before the registration. If the possible position values for the state are distributed over a finite region of space, we cannot think that the system in this state describes an object that is extended over this region in a similar way as a classical matter continuum would be extended. Indeed, the corresponding “real parts” of such an object would then have to move with superluminal velocity during some registrations.

Finally, we consider application of quantum mechanics to objects for which classical models are good approximations. As the basis of our approach to the issue of Universality, the third Trial Hypothesis reads:

Trial Hypothesis 1.3 *Every object has a quantum model that accounts for all its known physical properties.*

This does not require that each quantum model of the object must explain all its observed properties. We just assume that such models are in principle possible.

A particular application of TH 1.3 is that a physical object can have both a classical model and a quantum model. This defines a relation of the two models. Moreover, the classical model can be considered, in the sense of Minimum Interpretation, as a preparation as well as a registration apparatus for the quantum model. This is a very important observation that will be used in the remaining sections.

The three Trial Hypotheses 1.1, 1.2 and 1.3 and their consequences will strongly transform the Minimum Interpretation picture of quantum mechanics although the resulting theory can still be used to calculate the probabilities of outcomes of registrations following certain preparation by the methods provided by the standard quantum mechanics. But now it will be a theory that describes non-relativistic physical properties of objects including microscopic ones and the calculation of probabilities will be based on constructions of models of such objects.

Trial hypotheses 1.1 and 1.3 imply an atomistic picture of the world, with only few types of “atoms” that are present in huge numbers. The Hypotheses are, however, just starting points of a serious work. We must remove various paradoxes and problems that might result from them. This will be done in Chapters 2, 4 and 5.

1.3 Spin systems

In this and the next sections, the heuristic idea that properties are objective if they are uniquely defined by preparations will be specified and a list of objective properties will be given. The space of states will be described and the notions of proper mixture, of ontic and of epistemic states introduced.

Consider first a spin system. This is described by the two-dimensional Hilbert space \mathbf{H} carrying a unitary representation of the central extension $SU(2)$ of the proper rotation group $SO(3)$ (see e.g. [1], Section 7.4). The relation between the two groups is defined by the so-called central homomorphism, the map $h_c : SU(2) \mapsto SO(3)$. The most important observables of this system are generators of $SU(2)$ and they are called spin components.

The space of all self-adjoint operators on \mathbf{H} will be denoted by $\mathbf{L}_r(\mathbf{H})$. It is a four-dimensional real linear space. If we choose an orthonormal basis $|1\rangle, |2\rangle$ of \mathbf{H} ,

then any $\mathbf{A} \in \mathbf{L}_r(\mathbf{H})$ is described by a 2×2 complex symmetric matrix. We can define coordinates a_0, a_1, a_2 and a_3 on $\mathbf{L}_r(\mathbf{H})$ by writing \mathbf{A} as follows:

$$\mathbf{A} = a_0 \mathbf{1} + \sum_k a_k \sigma_k ,$$

where

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

is the so-called Pauli basis.

The action $\mathbf{A} \mapsto \mathbf{U}\mathbf{A}\mathbf{U}^\dagger$ of $SU(2)$ on $\mathbf{L}_r(\mathbf{H})$ can then be expressed by the coordinates as follows:

$$\mathbf{U}\mathbf{A}\mathbf{U}^\dagger = a_0 \mathbf{1} + \sum_l \left(\sum_k \mathbf{O}_{lk} a_k \right) \sigma_l , \quad (1.1)$$

the relation valid for any $\mathbf{A} \in \mathbf{L}_r(\mathbf{H})$, where $\mathbf{O} = h_c(\mathbf{U})$ and a_0, a_1, a_2, a_3 are the components of \mathbf{A} in the Pauli basis. Thus, the representation of $SO(3)$ on $\mathbf{L}_r(\mathbf{H})$ is not irreducible. It consists of a trivial representation on the subspace $a_1 = a_2 = a_3 = 0$ and the standard representation by the orthogonal 3×3 -matrices on the three-dimensional subspace $a_0 = 0$.

Each element of $\mathbf{L}_r(\mathbf{H})$ can be considered as an observable of the spin system. The positive elements with trace 1 can be considered as states of the system. Let us denote by $\mathbf{T}(\mathbf{H})$ the set of states. Clearly, $\mathbf{T}(\mathbf{H})$ is invariant with respect to the action of $SU(2)$, so that $SU(2)$ also acts on it.

Let us determine which subset of $\mathbf{L}_r(\mathbf{H})$ the space of states is. The decomposition of \mathbf{T} into Pauli basis defines its components t_j , $j = 0, 1, 2, 3$. Then the positivity conditions can be expressed by:

$$t_1^2 + t_2^2 + t_3^2 \leq 1/4 ,$$

and the trace condition by

$$t_0 = 1/2 .$$

Thus, $\mathbf{T}(\mathbf{H})$ can be identified with a three-dimensional disk of radius $1/2$ with centre $(1/2, 0, 0, 0)$ lying in the hyperplane $t_0 = 1/2$ in $\mathbf{L}_r(\mathbf{H})$.

The most interesting states lie at the boundary of the disk and are called *extremal* (more often “pure”). Let us study these states. In terms of the components in the Pauli basis, the matrix $\mathbf{M}_\mathbf{T}$ of \mathbf{T} is

$$\mathbf{M}_\mathbf{T} = \begin{pmatrix} 1/2 + t_3 & t_1 - it_2 \\ t_1 + it_2 & 1/2 - t_3 \end{pmatrix} ,$$

The determinant is easily calculated:

$$\det(\mathbf{T}) = 1/4 - (t_1^2 + t_2^2 + t_3^2) .$$

Hence, the boundary points have zero determinant. As the determinant of matrix $\mathbf{M}_\mathbf{T}$ is the product of its eigenvalues and the trace their sum, one of their eigenvalues is zero while the other is 1. Let us denote the normalised eigenvector of \mathbf{T} to the eigenvalue 1 by $|T\rangle$. Then

$$\mathbf{T} = |T\rangle\langle T| .$$

It is an important result: every extremal state is a projection onto a one-dimensional subspace of \mathbf{H} and vice versa.

$\mathbf{T}(\mathbf{H})$ is not a linear space, but it is a convex subset of $\mathbf{L}_r(\mathbf{H})$. If $\mathbf{T}_1, \mathbf{T}_2$ are two operators, then their convex combination is defined as

$$c_1 \mathbf{T}_1 + c_2 \mathbf{T}_2 \tag{1.2}$$

with $c_1 \geq 0$, $c_2 \geq 0$ and $c_1 + c_2 = 1$. From the definition of $\mathbf{T}(\mathbf{H})$ it follows that $\mathbf{T} = c_1 \mathbf{T}_1 + c_2 \mathbf{T}_2$ lies in $\mathbf{T}(\mathbf{H})$ if \mathbf{T}_1 and \mathbf{T}_2 do.

Convex combination $\mathbf{T} = c_1 \mathbf{T}_1 + c_2 \mathbf{T}_2$ can be represented by a line segment in the three-dimensional disk $\mathbf{T}(\mathbf{H})$ with end points \mathbf{T}_1 and \mathbf{T}_2 . The point corresponding to \mathbf{T} lies then between the end points so that it divides the line segment in the ratio $c_1 : c_2$. This follows from the fact that the linear structure of $\mathbf{L}_r(\mathbf{H})$ is represented by that of \mathbb{R}^4 .

Using this geometric picture, we can see that the boundary points cannot be written as convex combinations of other states. On the other hand, any internal point \mathbf{T} of $\mathbf{T}(\mathbf{H})$ can be written as convex combinations with any state \mathbf{T}_1 as the first component. Indeed, just draw a line through \mathbf{T}_1 and \mathbf{T} and choose any point \mathbf{T}_2 at this line that lies farther from \mathbf{T}_1 than \mathbf{T} .

In particular, if \mathbf{T}_1 is extremal (lies at the boundary), then \mathbf{T}_2 can also be chosen at the boundary, simply where the line intersect the boundary for the second time. Thus, each state inside $\mathbf{T}(\mathbf{H})$ can be written in infinitely many ways as a convex combination of extremal states.

However, from another point of view, any state \mathbf{T} that is different from the centre of the disk (proportional to the unit operator and called “completely chaotic state”) defines a unique decomposition into a convex combination of orthogonal extremal states. Indeed, it is just the spectral decomposition of \mathbf{T} ,

$$\mathbf{T} = t_1 \Pi_1 + t_2 \Pi_2 , \tag{1.3}$$

where t_1 and t_2 are the eigenvalues of \mathbf{T} while Π_1 and Π_2 are projections onto the eigenspaces. The state at the centre of the sphere can be written as:

$$\frac{1}{2} \mathbf{1} = \frac{1}{2} |\psi_1\rangle\langle\psi_1| + \frac{1}{2} |\psi_2\rangle\langle\psi_2| ,$$

where $\{|\psi_1\rangle, |\psi_2\rangle\}$ is any basis of \mathbf{H} .

The interior of the disk $\mathbf{T}(\mathbf{H})$ and each point of the the boundary are examples of the so-called faces of the convex set $\mathbf{T}(\mathbf{H})$ (for a general definition of faces, see , [51], Chapter III, Section 6 (p. 75)). Roughly, a face of $\mathbf{T}(\mathbf{H})$ is an intersection of $\mathbf{T}(\mathbf{H})$ with a hyperplane $\mathbf{V} \in \mathbf{L}_r(\mathbf{H})$ that satisfied the condition: if a point \mathbf{T} of $\mathbf{T}(\mathbf{H}) \cap \mathbf{V}$ can be written as a convex combination of $\mathbf{T}_1, \mathbf{T}_2 \in \mathbf{T}(\mathbf{H})$ then $\mathbf{T}_1, \mathbf{T}_2 \in \mathbf{T}(\mathbf{H}) \cap \mathbf{V}$. Thus, extremal points form zero-dimensional faces of $\mathbf{T}(\mathbf{H})$ and the interior of $\mathbf{T}(\mathbf{H})$ itself is a three-dimensional face. In our case, the hyperplanes can be only three- and zero-dimensional.

Observe that the geometry of the faces reflects the geometry of the Hilbert space as follows. The three-dimensional face contains states associated with the total Hilbert space while the zero-dimensional faces contain states associated with the one-dimensional subspaces of the Hilbert space. This will be a general picture. The association is defined as follows. Let \mathbf{H}_1 be a subspace of \mathbf{H} and Π_1 be the projection onto \mathbf{H}_1 . State \mathbf{T} is associated with \mathbf{H}_1 if

$$\mathbf{T} = \Pi_1 \mathbf{T} \Pi_1 .$$

It follows that $\mathbf{T} \in \mathbf{T}(\mathbf{H}_1)$.

Let us now turn to the question of how observables can be measured and state prepared. Stern and Gerlach [69] have measured the spin components by observing the deflection of a neutral beam of silver atoms in an inhomogeneous magnetic field in 1922. We shall use the account of an idealised experiment of this kind given in [58].

Silver atoms evaporate in an oven and pass through a velocity selector. Let us call this preparation \mathcal{P} . To be registered, the resulting beam crosses an inhomogeneous magnetic field in a magnet apparatus \mathcal{M} and, finally, strikes a photo plate perpendicular to the direction of the beam. All the impacts are then found in two narrow equally dense strips. Let \vec{n} be one of the two unit vectors parallel to the plate and perpendicular to the strips. Vector \vec{n} depends on the orientation of the magnet. The two strips would be rotated by the angle by which the magnet \mathcal{M} in the plane perpendicular to the beam were. In this way, vector \vec{n} also indicates the orientation of the magnet, that is the orientation of the Stern-gerlach apparatus. Then, a registration by Stern-Gerlach apparatus with orientation \vec{n} can have only two outcomes: the atom hits either the upper or the lower strip, where “upper” and “lower” are defined by \vec{n} so that the upper strip lies in the direction of \vec{n} from the lower one.

Let us choose a Cartesian coordinate frame (x_1, x_2, x_3) and describe the orientation of the magnet by the components of a unit vector \vec{n} in this frame. Then the observable that is registered by the apparatus with orientation \vec{n} is the spin in

direction \vec{n} . It is described by the operator

$$S(\vec{n}) = \frac{1}{2}\hbar(n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3) ,$$

where σ_k is the k -th Pauli matrix. The spin with orientation of the third axis has eigenvalues $\pm\hbar/2$ and eigenvectors

$$|\uparrow\rangle = (1, 0) \quad |\downarrow\rangle = (0, 1) .$$

In this way, the basis of \mathbf{H} in which Pauli matrices are defined is associated with the frame (x_1, x_2, x_3) and relation (1.1) implies that rotation of a Stern-Gerlach apparatus by \mathbf{O} corresponds to the $SU(2)$ transformation of the registered observable.

A Stern-Gerlach apparatus can also be used for preparations. Let preparation $\mathcal{P}(\vec{n})$ be defined as follows. The first part of it is as above with the magnet \mathcal{M} being set in direction \vec{n} . Then, instead of a detector, a screen is placed in the way of the beams coming from \mathcal{M} so that the lower beam is blocked off and the upper beam is let through.

Now, we can embark on the discussion of states. RCU interpretation of states as objective properties of individual systems enables us to introduce the concept of state statistics that is different and independent of the statistics of registration outcomes. To explain this point, we can use Newtonian mechanics. A probability distribution on the phase space of a Newtonian object can be called an *epistemic* state if the notion of Newtonian mechanics is adopted that a system always *is* at a definite point of the phase space. Then, the epistemic state expresses our incomplete knowledge of the system. A point of the phase space is an example of a state that provides a maximal possible information on the Newtonian object. States of this kind can be called *ontic*.

Let us now study the epistemic-or-ontic question for quantum states. Consider a preparation that uses two Stern-Gerlach apparatuses in a parallel arrangement. That is, the first of them performs a preparation $\mathcal{P}(\vec{n}_1)$ while the second does $\mathcal{P}(\vec{n}_2)$, and they are oriented in such a way that the beams created by them are approximately parallel but cross at some distance from the apparatuses. Let further the intensity of the beams differ so that they have rates, P_1 and P_2 , respectively, $P_1 \neq P_2$ and $P_1 + P_2 = 1$. The preparation of the state that results at the intersection of the beams is called *statistical mixture* $P_1\mathcal{P}(\vec{n}_1) + P_2\mathcal{P}(\vec{n}_2)$ of preparations $\mathcal{P}(\vec{n}_1)$ and $\mathcal{P}(\vec{n}_2)$.

A general definition of statistical mixture of preparations is as follows:

Definition 1.2 Let \mathcal{P}_1 and \mathcal{P}_2 be two preparation of S and $P \in [0, 1]$. Statistical mixture

$$\mathcal{P} = P\mathcal{P}_1 + (1 - P)\mathcal{P}_2 \tag{1.4}$$

of the two preparations is a new preparation constructed as follows. Let object \mathcal{S} be prepared either by \mathcal{P}_1 or by \mathcal{P}_2 in a random way so that \mathcal{P}_1 is applied with probability P_1 and \mathcal{P}_2 with probability $1 - P_1$.

This definition can easily be extended to any number of preparations. The preparation of the state at the intersection of the beams is not completely known, but some knowledge about it is available (namely that it prepares two known states, each with a known rate).

Now, we can formulate a standard assumption of quantum mechanics.

Assumption 1.1 *Let preparations \mathcal{P}_1 and \mathcal{P}_2 preparing states T_1 and T_2 of object \mathcal{S} are mixed with rates P_1 and P_2 . The state prepared in this way is*

$$P_1 T_1 + P_2 T_2 . \quad (1.5)$$

It is called proper mixture of states T_1 and T_2 .

The generalisation of this assumption to any quantum system is easy. In literature, there are various names proposed for such a state: *direct mixture* [51], *proper mixture* [13] or *Gemenge* [9].

It follows that a convex combination (1.2) can also have a physical meaning: it can be a proper mixture of states T_1 and T_2 . Suppose that state T is a such a proper mixture. Then it is not an extremal state and hence decomposable into an infinite number of non-equivalent convex combination. Only one of them has then a physical meaning of proper mixture. The others are just mathematical possibilities of how T can be written. The proper mixture does not differ from all other convex combination by any mathematical property, only by the method of preparation. The expression (1.2) itself does not contain any information on this preparation. Thus, we find it useful, to introduce a special notation for those convex combination that are proper mixtures:

$$P_1 T_1 +_s P_2 T_2 .$$

Proper mixtures are still states uniquely determined by their preparations and hence they are objective properties of the prepared systems. This is similar as in Newtonian mechanics. One can also understand it as follows. A proper mixture gives a direct information on an ensemble of equally prepared states and surely is an objective property of this ensemble. However, to be an element of a particular ensemble is then also an objective property for an individual subsystem of this ensemble.

For a state to be a proper mixture, there must be some observational reasons, such as special preparations or other observations of macroscopic objects. Even if, mathematically, there is a unique convex decomposition it would be wrong to

consider such a convex combination as a proper mixture. An example of such a unique decomposition is the spectral decomposition (see, e.g., [1], p. 20) of a state operator that has a non-degenerate spectrum. Indeed, state (1.5) can also have a unique spectral decomposition into extremal states but this decomposition does not correspond to the way the state has been prepared.

The possibility that a convex combination can represent a proper mixture can lure one into a conjecture that there is some deeper physical difference between extremal states, which cannot be written as a non-trivial convex compositions, and non-extremal states, which always can. Thus, extremal states are often called “pure” and non-extremal ones “mixtures”. However, a non-extremal state lies inside a three-dimensional face and can be written as convex combination in an infinity of different ways. If there is no further reason to choose a particular combination, there is no reason either to consider the state as a particular proper mixture. Such states are often called “improper mixtures”.

Quantum states are thus usually classified into extremal states, improper and proper mixtures. We assume now:

Trial Hypothesis 1.4 *Extremal states and improper mixtures are ontic, proper mixtures are epistemic, quantum states.*

TH 1.4 refines Trial Hypothesis 1.1 by stating that the quantum states that are not proper mixtures give a maximum information on quantum systems.

Observe that the notions of ontic and epistemic states we have introduced are different from how such notions are understood within Minimum Interpretation. Minimum Interpretation considers a whole measurement process, including its preparation and its registration, as a real entity. The prepared state and the registered value of an observable are objective properties of each individual measurement process. However, the state does not determine which value really occurs. In this sense, a state is always an epistemic description of a measuring process. This is of course also true for RCU interpretation but the state is then also an objective property of the prepared system, for which it can be an ontic description. The registered values become real only in registration. It is important to realise this difference to prevent misunderstanding.

Examples of three different preparations that illuminate the ideas about proper mixtures can be constructed for the system of two spins. Consider the system composite from an electron (index (2)) and a positron (index (1)) prepared in extremal state $T_- = |\psi_-\rangle\langle\psi_-|$, where

$$|\psi_-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle^{(1)} \otimes |\downarrow\rangle^{(2)} - |\downarrow\rangle^{(1)} \otimes |\uparrow\rangle^{(2)}) . \quad (1.6)$$

Subsystem states $|\uparrow\rangle^{(1)}, |\downarrow\rangle^{(1)}$ and $|\uparrow\rangle^{(2)}, |\downarrow\rangle^{(2)}$ define basis $|\uparrow\rangle^{(1)} \otimes |\uparrow\rangle^{(2)}, |\downarrow\rangle^{(1)} \otimes |\uparrow\rangle^{(2)}, |\uparrow\rangle^{(1)} \otimes |\downarrow\rangle^{(2)}, |\downarrow\rangle^{(1)} \otimes |\downarrow\rangle^{(2)}$ of $\mathbf{H} \otimes \mathbf{H}$.

Let Alice is going to register spins of the positron and Bob those of the electron by Stern-Gerlach meters. Let the first experiment be such that Alice does not do anything just leaving the positron to pass without registration. Then the state prepared in this way for Bob is the partial trace $\mathsf{T}^{(2)} = \text{tr}^{(1)}(\mathsf{T}_-)$, see [1], Section 8.3. An easy calculation gives:

$$\mathsf{T}^{(2)} = \frac{1}{2} |\uparrow\rangle^{(2)} \langle \uparrow|^{(2)} + \frac{1}{2} |\downarrow\rangle^{(2)} \langle \downarrow|^{(2)} ,$$

which has the following components in the basis of \mathbf{H} :

$$\mathsf{T}^{(2)} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} .$$

This is a completely random state so that the probability of obtaining value $+\hbar/2$ by registration of observable $\mathsf{S}^{(2)}(\vec{n})$ is $1/2$ independently of \vec{n} . Hence, $\mathsf{T}^{(2)}$ contains zero information on the spin of the electron. Still, it is an ontic state in the sense that it is not a proper mixture of states each of which contains any non-trivial information about the spin. If this were the case, then each individual electron arriving at Bob's detector would be objectively in some of these more specific states. But this would contradict the rule that objective states must be prepared. The studied example is thus an interesting case of ontic state.

In the second experiment, let Alice register $\mathsf{S}_3^{(1)}$ on each positron but do not record the outcomes. Each such registration creates one of the states

$$\mathsf{T}_{\uparrow}^{(2)} = |\uparrow\rangle^{(2)} \langle \uparrow|^{(2)} , \quad \mathsf{T}_{\downarrow}^{(2)} = |\downarrow\rangle^{(2)} \langle \downarrow|^{(2)}$$

with probability $1/2$ at Bob's laboratory. That is, the prepared state of the electron is now a proper mixture, $1/2\mathsf{T}_{\uparrow}^{(2)} +_s 1/2\mathsf{T}_{\downarrow}^{(2)}$. This preparation of state $\mathsf{T}^{(2)}$ is incompletely known by both Alice and Bob.

State T_- implies strong anticorrelations of spins $\mathsf{S}^{(1)}(\vec{n})$ and $\mathsf{S}^{(2)}(\vec{n})$ for any unit vector \vec{n} . The intuitive idea that a relation of $\mathsf{S}^{(2)}$ to another system can be considered as a property of $\mathsf{S}^{(2)}$ motivates the following claim: The above example shows that general rules of quantum mechanics imply the existence of properties of system $\mathsf{S}^{(2)}$ which, on the one hand, are prepared by the above preparation but, on the other, are not encoded in its state $\mathsf{T}^{(2)}$. The correlations between the spins of the electron and positron are due to their entanglement and are described e.g. in Section 20.2 of [1]. Observe that the correlations *are* determined by a state, namely T_- but it is a state of a composite system containing $\mathsf{S}^{(2)}$.

Hence, the first example suggests:

Trial Hypothesis 1.5 *Objective properties of a quantum system S can be divided into three classes: 1. structural properties of S , 2. a state of S and the properties*

determined uniquely by the state (such as expectation values of a fixed observable), 3. the properties of the state of a system that has been prepared so that it contains S as a subsystem if such properties concern S but are not determined by the state of S itself (such as the way S is entangled with other systems).

This will be a general hypothesis of RCU interpretation. It can also be considered as a rigorous definition of objective properties of quantum systems.

The fact that Bob's electron is in an ontic state in spite that it is totally random must be interpreted as mathematical expression of the inefficiency of the observables pertaining to an individual electron. If one registered simultaneously also an observable of the positron, some non-trivial information about the correlations between observables of the two systems can be obtained.

The correlations between the spins of the positron and the electron encoded in state T_- is e.g. a strong anticorrelation between the third components of the spins. Thus, if Alice sees $+\hbar/2$, Bob must see $-\hbar/2$. It is well-known that some kind of nonlocality is implied by that. The registration apparatuses can be arbitrarily far away from each other at the time of their simultaneous registrations. The electron spin value is determined when the positron spin is registered, but not earlier. But the positron spin is also only determined when it is registered. To be correlated with the outcome for the electron, it must “know” the outcome for the electron. This holds independently of whether we limit ourselves to the Minimum Interpretation or add any assumption about reality of states, because the correlations concern primarily outcomes of registrations, that is, states of classical apparatuses. Within the Minimum Interpretation, the nonlocality concerns spooky communication between remote meters, while within our interpretation, the state of the composed object is a real “thing” extended over two regions of the space that are far away from each other and it must change abruptly in both regions if a registration is made in just one of them. But the extension is “abstract” in the sense that it is not due to different real (material) local subsystems as the extension of a classical continuum is.

1.4 Particles

Particles are quantum systems associated with the Hilbert space \mathbf{H}_s constructed from the complex linear space of functions $\psi(\vec{x}, m)$, $\vec{x} \in \mathbb{R}^3$, $m = -s, \dots, s$ and $s = 0, 1/2, 1, \dots$. The Hilbert space carries an irreducible unitary representation of central extension \bar{G}_μ^+ of proper Galilean group G^+ . The central extension depends on the mass μ of the particle (for details, see Chapter XII, Section 8 of [75]).

The most important difference between the spin and particle systems is that the space and time aspects of quantum mechanics are missing for the former and fully

accounted by the latter. This allows a further development of the ideas introduced in the previous section.

In Section 1.3, a special sign, $+_s$, for proper mixing of states has been introduced. Now, we can discuss the proper mixing in more detail. First, as a mathematical operation, $+_s$ is commutative and associative:

$$P_1 T_1 +_s P_2 T_2 = P_2 T_2 +_s P_1 T_1$$

for any two states T_1, T_2 and rates P_1, P_2 , and

$$(P_1 T_1 +_s P_2 T_2) +_s P_3 T_3 = P_1 T_1 +_s (P_2 T_2 +_s P_3 T_3)$$

for any three states T_1, T_2, T_3 and rates P_1, P_2, P_3 . This follows directly from the definition.

Second, the definition also directly implies that proper mixing is independent of representation and invariant with respect to transformation by Galilean group,

$$U(P_1 T_1 +_s P_2 T_2) U^\dagger = P_1 U T_1 U^\dagger +_s P_2 U T_2 U^\dagger ,$$

where U may be both the unitary transformation between representations and transformation by a representative of an element of Galilean group. In particular, a proper mixture remains a proper mixture during time evolution.

Finally, the invariance of the proper mixture with respect to system composition follows also from the definition of $+_s$ [36]:

Composition Invariance of Proper Mixture *Let the state of composed system $S^{(1)} + S^{(1)}$ be T . The necessary and sufficient condition for state $tr^{(2)}(T)$ of $S^{(1)}$ to be a proper mixture,*

$$tr^{(2)}(T) = P_1 T_1^{(1)} +_s P_2 T_2^{(1)} ,$$

where $T_1^{(1)}$ and $T_2^{(1)}$ are some states of $S^{(1)}$, is that T itself is a proper mixture of the form

$$T = P_1 T_1^{(1)} \otimes T_1^{(2)} +_s P_2 T_2^{(1)} \otimes T_2^{(2)} ,$$

where $T_1^{(2)}$ and $T_2^{(2)}$ are some states of $S^{(2)}$.

The question of whether the states $w T_1 +_s (1 - w) T_2$ and $w T_1 + (1 - w) T_2$ of object \mathcal{S} can be distinguished by registrations is interesting and important. State operator T does not, by itself, determine the statistical decomposition of a prepared state described by it, unless T is extremal so that its every convex decomposition is trivial. Registrations that are limited to observables of S cannot distinguish proper and improper mixtures because the registration probabilities depend only on the state operator.

However, if observables of composite systems containing S are also admitted, then the difference of the two states can be found by measurements, for instance if the state of the composite system is extremal and Composition Invariance of Proper Mixture holds. Another aspect of the distinction is the following. If one part of the studied system is a macroscopic object with both classical and quantum models such as a registration apparatus, and if the state of the whole system is a convex combination of two quantum states each of them associated with a different classical state of the macroscopic object, then such a convex combination must be a proper mixture because the macroscopic object is always only in one of the two classical states. We shall return to this in Chapter 5.

Extremal states allow another mathematical operation, a linear superposition, which is different from a convex combination. If

$$|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle ,$$

the corresponding state operator is

$$|\psi\rangle\langle\psi| = |c_1|^2|\psi_1\rangle\langle\psi_1| + c_1c_2^*|\psi_1\rangle\langle\psi_2| + c_1^*c_2|\psi_2\rangle\langle\psi_1| + |c_2|^2|\psi_2\rangle\langle\psi_2| . \quad (1.7)$$

This differs from the convex combination of the two states,

$$\mathbf{T} = |c_1|^2|\psi_1\rangle\langle\psi_1| + |c_2|^2|\psi_2\rangle\langle\psi_2| ,$$

by the non-diagonal (cross) terms. The difference can be revealed by the registration of suitable observables. For example, expectation values of observable $\mathbf{O} = |\psi_1\rangle\langle\psi_2| + |\psi_2\rangle\langle\psi_1|$, for orthogonal states $|\psi_1\rangle$ and $|\psi_2\rangle$, are: $\langle\psi|\mathbf{O}|\psi\rangle = c_1^*c_2 + c_2^*c_1$ and $\text{tr}(\mathbf{O}\mathbf{T}) = 0$. The non-diagonal terms also describe correlations that can be revealed by registrations. The cross terms lead also to interference phenomena (such as the electron interference in interference experiments, e.g. [73]).

If a preparation is not completely known, we can still assume that it prepares a state described by some state operator of which we do not know whether it is a proper or improper mixture. In many cases, the structure of a possible proper mixture is not important because sufficiently many properties of the state are independent of it and everything one needs can be obtained from the state operator.

It may be helpful to compare quantum states of our interpretation with states of Newtonian mechanics. Let us define a state of an arbitrary Newtonian system as a point of the (many-dimensional) phase space $\mathbf{\Gamma}$ of the system. Newtonian state defined in this way is generally assumed to satisfy:

1. *objectivity*: a state of a system is an objective property of the system,
2. *generality*: any system is always in some state,

3. *exclusivity*: a system cannot be in two different states simultaneously,
4. *completeness*: any state of a system determines the values of all observables that can be measured on the system,
5. *separability*: the state of a composite system is determined by the states of its subsystems,
6. *locality* the state of a system determines the space position of the system, that is positions of all its subsystems.

An incomplete information about the state of a system can be described by a probability distributions on $\mathbf{\Gamma}$. Indeed, because of the generality, the system always is at a particular point of $\mathbf{\Gamma}$, but we do not know at which. Such a distribution is sometimes called *statistical state*. In any case, we distinguish a state from a statistical state.

As was mentioned in Section 1.3, statistical states can be called epistemic and points of $\mathbf{\Gamma}$ can be called ontic. This distinction depends clearly on hypotheses about what can and what cannot be known¹.

In quantum mechanics, we have the following picture: The space of quantum ontic states is $\mathbf{T}(\mathbf{H}_s)$ (more details about the structure of this space can be found in Chapter III, Section 6 of [51]; an example of it for the two-dimensional Hilbert space is described in the section on the spin system). Epistemic quantum states—proper mixtures—can be described by probability distributions on $\mathbf{T}(\mathbf{H}_s)$.

Here, a “state of a system” is interpreted as a state that is prepared for the system. Then, point 1., 2. and 3. are also valid for quantum mechanics. Point 4. does not hold in quantum mechanics because each ontic state of a system determines only probability distributions of values of its observables. Still, the information given by an ontic state is maximal in the following sense. If the same ontic states \mathbf{T} is repeatedly prepared then each element of the ensemble of systems obtained in this way cannot be considered to be in another state than \mathbf{T} . That an improper mixture can give a maximum information exactly as a pure state does is also a consequence of Composition Invariance of Proper Mixture. Indeed, let, in Composition Invariance of Proper Mixture, \mathbf{T} be extremal and $\mathbf{T}^{(1)}$ be not. If there is more information to have on system $S^{(1)}$ than that contained state $\mathbf{T}^{(1)}$ then there is also more information on the composed system S than state \mathbf{T} provides. Indeed, state \mathbf{T} is to give a maximum information on S and thus on $S^{(1)}$ and $\mathbf{T}^{(1)}$ gives as much information on $S^{(1)}$ as \mathbf{T} does.

¹Of course, this is based on the hypotheses that point-like states really exist. The assumption of real existence of phase space points is, however, surely incorrect, if quantum mechanics is valid. We shall study these questions in more detail in Chapter 2.

As we have seen in the previous section, Point 5 does not hold in quantum mechanics. Finally, the position of a subsystem is an observable and it has, therefore, no predetermined value if the ontic state of a composite system is known. Hence, Point 6. is also wrong in quantum mechanics.

The properties that distinguish quantum objects from each other have been classified and listed by TH 1.5. The arguments and motivation for this TH have been given there for a two-spin system. Analogous arguments and motivations are valid for general systems defined in the present section.

Another difference between quantum and Newtonian mechanics is that a proper mixture can be mathematically represented by the same state operator as an ontic state while Newtonian ontic and epistemic states are always represented by different mathematical entities.

Chapter 2

Maximum-entropy packets

According to Trial Hypothesis 1.3, the physical objects that have classical models also admit quantum models. The subject of the present chapter is the questions of what the relation between the classical and the quantum model of an object is, how such quantum models can be constructed and how the classical properties can be derived from them.

A well-known problem of quantum theory of classical properties is that some obvious features of the classical models seem incompatible with quantum mechanics. Let us list the most important features of this kind.

1. Every classical system is always only in one of its possible classical states independently of whether it is observed or not.
2. Every classical system possesses all its properties independently of whether it is observed or not. Classical properties are objective.
3. Classical systems are durable, that is they do not suddenly appear or disappear except in very special cases.
4. Classical properties (including states) are robust, that is, a classical measurement can be done in such a way that the state of the measured object is arbitrarily weakly disturbed.

One important consequence of Point 2 and some properties of classical models is the existence of *sharp trajectories*: sharp values of classical observables, such as position, momentum, field strengths, charge current, as well as temperature and internal energy of phenomenological thermodynamics, can be ascribed to the objects independently of whether they are observed or not. Another one is that of *no superpositions*: Nobody has ever seen a chair, say, to be in a linear superposition of being simultaneously in the kitchen as well as in the bedroom.

Hence, we shall have at least the following difficulties.

- A According to Trial Hypothesis 1.2, values of quantum observables are only created by registrations. Then Point 2 would lead to a problem, if classical observables were too closely related to quantum ones.
- B The basic classical assumption of sharp trajectories do not seem easily compatible with the Heisenberg uncertainty relations.
- C Attempts to get quantum trajectories as sharp as possible lead to states with minimum uncertainty. But the quantum states of minimum uncertainty are extremal and such states can be linearly superposed.
- D Quantum states are not disturbed only by measurements of very few very special observables (for more discussion, see Refs. [47, 9]). Especially, the extremal states are rather fragile. Then Point 4 would lead to a problem if classical states are too closely related to quantum ones.

There are various proposals of how at least some of these difficulties could be dealt with. For example, one assumes that some phenomena exist at the macroscopic level which are not compatible with standard quantum mechanics. Such phenomena may prevent linear superpositions of extremal states (see, e.g., [47] and the references therein). Or they can lead to a spontaneous dissipation of extremal states. The dynamical collapse theory (see, e.g., [19]) is of this kind. The second example are theories based on the idea that certain kind of coarse-grained operators [41, 60, 44] associated with macroscopic systems are measurable but fine grained are not. The third example are the Coleman-Hepp theory [38, 2, 4] and its modifications [67, 61, 76]: they are based on some specific theorems that hold only for infinite quantum systems (see the analysis in [2]) or for asymptotic regions [76]. Other examples assume that the macroscopic realism is only apparent in the sense that there *are* linear superpositions of macroscopic states but the corresponding correlations are difficult or impossible to observe. For example, the quantum decoherence theory [22, 66, 81] works only if certain observables concerning both the environment and the quantum system cannot be measured (see the analysis in [13, 8]). Here, also the so-called modular interpretations belong ([8]).

Admittedly, the list is too concise and rather incomplete. Its only purpose is to suggest that there *are* problems and there is a vast literature on them. However, the aim of this chapter is to focus on our approach, which is new and very different. It rejects sharp trajectories as an idealisation and is limited to looking for quantum derivation of only those classical properties that are themselves fuzzy. This opens a way to an application of statistical methods. A motivation of such an approach is that some classical properties have already been successfully derived from quantum mechanics: the properties are the thermodynamic ones and the method is that of

quantum statistical thermodynamics. We are going to generalise statistical methods to Newtonian mechanics.

2.1 Hypothesis of High-Entropy States

To see how thermodynamic properties are derived, look at a vessel of gas in a laboratory. Let us denote the gas object by \mathcal{S} . First, we describe a classical model S_c of it. Let the gas be in thermodynamic equilibrium, the volume of the vessel be Ω , the gas pressure be P , its temperature be T and its mass M . As for the chemical composition, let the gas be the monoatomic helium. We can calculate various thermodynamic quantities using formulas of phenomenological thermodynamics. For example, the number of molecules N is given by MN_A/M_{mol} , where N_A is the Avogadro number and M_{mol} is the molecular weight, while the internal energy E is given by

$$E = \frac{3}{2}kNT ,$$

where k is the Boltzmann constant. In this way, the classical model S_c of \mathcal{S} is defined.

Second, let us construct a quantum model S_q of \mathcal{S} : the system of N spin-zero point particles, each with mass $\mu = M_{\text{mol}}$, in a deep potential well of volume Ω with Hamiltonian

$$\mathbf{H} = \sum_{k=1}^N \frac{|\vec{\mathbf{p}}^{(k)}|^2}{2\mu} ,$$

where $\vec{\mathbf{p}}^{(k)}$ is the momentum of k -th particle in the rest system of Ω . \mathbf{H} is simultaneously the operator of the internal energy of S_q (total energy in a rest frame).

An important assumption of the quantum model is the choice of state. It is the state that maximises the von Neumann entropy¹ for fixed value E of the expectation value of the internal energy. Such state is called *Gibbs state*. All properties of S_c can then be calculated from S_q as properties of the Gibbs states.

We can of course see that both models S_c and S_q are incomplete pictures of object \mathcal{S} . This is a general property of models. Still, we have a classical system and a quantum system and a definite relation between the two: they ought to refer to one and the same object. This relation can be made deeper if we realise that the classical system S_c can play the role of a preparation apparatus for S_q or, at least, an essential part of such an apparatus. Then, the system S_q and its quantum state are determined by S_c and its classical state. Moreover, S_c can also play the role of a meter for S_q . Indeed, by observing a classical property of S_c , we obtain an information on S_q . For instance, if we (macroscopically) isolate the vessel and

¹For definition, see e.g. Section 9-1 of [58].

keep it so for some time the gas settles in a state of thermodynamic equilibrium. This is achieved with the help of some further (macroscopic) tools different from the system S_c alone. Similarly, if we measure the temperature of the S_c , we use a classical system, the thermometer, that is different from S_c . However, S_c itself must take part in these procedures and as it is “at the classical side” of the experiments, it is an important part of the preparation and registration of S_q .

We also know that many classical properties of classical systems can be observed directly via our senses, that is without mediation of any other registration devices and that such observations do not disturb the observed classical system. As mentioned at the beginning of the present chapter, this is a feature of classical systems and we shall try to explain it by the quantum mechanics of the corresponding quantum model.

The thermodynamic example motivates a general trial hypothesis about a relation between classical and quantum models of the same object. The assumption works at least in the case of thermodynamics and it can be formulated as follows.

Trial Hypothesis 2.1 *Let S_c be a classical model and S_q a quantum model of object \mathcal{S} . Then S_c can be considered as an essential part of the preparation device and, simultaneously, as an essential part of a meter, for the quantum model S_q . The meter in question registers values distinguishing the quantum states of S_q that are associated with different classical states of S_c .*

The main principle of our theory is a generalisation of the Gibbs-state idea to all classical properties, including the mechanical ones. Thus, we supplement TH 2.1 by the following trial hypothesis:

Trial Hypothesis 2.2 *Let a real object \mathcal{S} has a classical model S_c and a quantum model S_q . Then all properties of S_c are selected properties of some high-entropy states of S_q .*

This hypothesis is only possible if some of already stated trial hypotheses hold true, such as TH 1.3 and 2.1. Hypothesis 2.2 is a heuristic one and it is therefore formulated a little vaguely. It will be made clearer by examples of its use studied in this chapter. But some examples can make it clearer already now.

Consider first states of macroscopic systems that are at or close to absolute zero of temperature. These are approximately or exactly extremal and maximize entropy at the same time but the entropy, though it could be maximal, is not high. We are not going to consider these objects as classical. Second, consider states of macroscopic systems at room temperature that are not at their thermodynamic equilibrium but are close to it. There are many such states and they and the systems can be described by classical physics to a good approximation. They are not in maximum- but in

high-entropy states. As the final example, consider a mechanical watch such as an old reliable Swiss chronometer. It is composed of a great number of small mechanical subsystems, cogs, shafts, bearings, etc., that are ordered according to an ingenious plan, so that the state of the system composed of such subsystems as units can be considered to have zero entropy. However, each such subsystem is macroscopic and its internal state is close to thermal equilibrium that is to the state of maximum entropy. The loss of entropy due to the order of the mechanical subsystems is negligible with respect to the sum of their internal entropies. Hence, the watch can also be considered as being in a high-entropy state.

An important advantage of TH's 2.1 and 2.2 is that they suggest ways in which the problems mentioned at the start of this chapter can be solved. Problem A could be approached as follows. Our theory of objective properties of quantum systems in Sections 1.2 (TH 1.1), 1.3 and 1.4 justifies the assumption that quantum states, even the high-entropy ones, are objective. If classical properties are properties of some states of the quantum model, they will also be objective. For example, the classical internal energy and pressure can be assumed to be the expectation values of some quantum observables in a high-entropy state. Also, there are many classical properties of real objects that have been successfully modelled by quantum mechanics, such as temperature, entropy, electrical conductivity or specific heats that are not expectation values of quantum observables. They are still properties of some high-entropy states.

Problem C is connected with the assumption that classical systems are modelled on coherent states of the corresponding quantum model and with the fact that the coherent states are extremal. If we however look at any object of our everyday experience, such as a chair, we can immediately see that, as a quantum system, it cannot be in an extremal state: it is near its thermodynamic equilibrium at the room temperature. In principle, it might be possible to prepare a macroscopic quantum system in an extremal state, but it is fiendishly difficult. One had to bring its entropy to zero for that! Hence, not only is its state a high-entropy one, but it is also no proper mixture of extremal states. According to Assumption 2.3.5, existence of such a proper mixture could only be justified if each of the extremal states were extra prepared. For the above reasons, such a preparation is practically impossible.

As for Problem D, we first observe that any classical state in our sense is a state of a classical model. For example, in the above example of the ideal gas, the classical state is determined by values of three quantities: the volume, the particle number and the internal energy. From the quantum point of view, the volume is an external field while the energy and particle number are expectation values of the internal energy and the particle number operators. Although the three values determine the classical state uniquely, many different quantum states are compatible with the same

three values. Then, even if quantum states may be disturbed by an observation, the corresponding classical states need not be. This idea of a conceptual difference between quantum and classical state is essential for our theory. For example, in the theory of Leggett-Garg inequality [48], such a difference is ignored.

Finally, Problem B is solved in phenomenological thermodynamics by accepting that the sharpness is only an idealisation or approximation and the “sharp” values are in fact fuzzy.

Of course, even if such project of constructing quantum models corresponding to classical ones worked nicely the question would remain open of what is the origin of all the ontic high-entropy states that are observed in such a great abundance around us. The physical foundations of thermodynamics are not yet completely understood but there are many ideas around about the origin of high-entropy states. Their existence might follow partially from logic (Bayesian approach, [40]) and partially from quantum mechanics (thermodynamic limit, [72], Vol. 4). Some very interesting models of how maximum entropy quantum states come into being are based on entanglement [16, 59, 50, 23]). In particular, in [16] a spontaneous evolution to such a state of a system S_q is proved. The main premises of the proof are that S_q is a subsystems of S'_q that S'_q is in an extremal state and that there is a weak interaction between S_q and the rest of S'_q . It then would follow from our analysis in Section 1.3 that the state of S_q could not be a proper mixture at any time. However, even if we do not know the physical cause of high-entropy states, we can just try to construct quantum models of classical properties and the high-entropy states can be used as one of the assumptions without really understanding their origin.

An understanding of classical macroscopic world mediated by quantum mechanics would not be complete without a quantum theory of classical measuring apparatuses. It seems that the final stage of many classical measurements is a sight by a human eye. The pictures that are formed with the help of eyes are relatively smooth functions on a two-dimensional plane representing the intensities of light of different small frequency intervals.

From the quantum point of view, they are apparently results of a huge number of photons registered simultaneously and not decomposed into the single photon events. The retina contains millions of cone cells and each such cell can give its signal only if a number of photons, as a rule tens to hundreds, simultaneously or at least in short intervals after each other, hit the cell so that a sufficiently high potential is built in it. Thus, the eye accumulates and synthesise the individual photon registrations while an ideal quantum measurement apparatus must be able to detect and distinguish individual photons.

Similarly, a photo-emulsion works. An emulsion consists of millions of silver-halogen corns, tiny crystals of diameter about 100 nm. Each corn must be hit

by at least four photons so that pure silver impurities form in the crystal. These impurities are condensation centres around which the developer causes a change in the structure of the corn from silver-halogen to pure silver. Again, a continuous field of intensities of different colours is the result, which is not decomposed into individual photon registrations. If we know the bulk intensities around each point and the energies of the photons of the corresponding colour, we could do the analysis and find approximately the intensity and colour distribution over the picture from which the quantum probabilities of the captured photons could be established.

The light that is registered by these apparatuses in the described way is what is called in optics “incoherent light”: clouds of many photons in a state of high entropy. It seems therefore, that classical measuring apparatuses are quantum measuring apparatuses that are deficient in the sense that they only register if clouds of a large number of particles arrives at them. The apparatuses are constructed in such a way that the results of their registrations are associated with probability distributions of quantum values. On the one hand, the probability distributions can be considered as properties of the registered quantum states. On the other, the apparatus can then be considered as registering directly a classical property. We shall call such a measurement a *cumulative measurement*. These ideas are not new. We find, e.g., in [55], p. 4:

...the idea that the principle of continuity (“natura non facit saltus”), prevailing in the perceived macroscopic world, is merely simulated by an averaging process in a world which in truth is discontinuous by its very nature. This simulation is such that man generally perceives the sum of many billions of elementary processes simultaneously, so that the levelling law of large numbers completely obscures the real nature of the individual processes.

The registrations of the eye or the camera are relatively simply related to the similar quantum registrations. However, many classical measurements are less directly related to registrations of quantum observables. Examples are measurements of the temperature, of the heat capacities or of the pressure. Still, we assume that even such classical measurements ultimately give information about some quantum observables.

2.2 High-entropy states in Newtonian mechanics

There seems to be a difficulty with TH 2.2. For a body such as the chair, the Galilean invariance of quantum theory leads to the separation of the bulk motion from all other degrees of freedom. The motion of mass centre and of the total angular

momentum with respect to the mass centre is described by Newtonian mechanics. It comprises only six degrees of freedom while statistical methods seem to show their full power for systems with a huge number of weakly coupled degrees of freedom.

Let us look more closely at Newtonian mechanics. One of its basic hypotheses is that any system at any time is objectively at some point of its phase space. The time dependence of this point forms a trajectory, a curve in the phase space. Let us call this *Sharp Trajectory Hypothesis* (STH). In Newtonian mechanics, states may be more general than points of phase space: probability distributions on the phase space are also viewed as states. The points have been called ontic, the non-trivial distributions epistemic states in Section 1.3. This is justified by STH: a non-trivial probability distribution describes the (incomplete) state of our knowledge on the system. As already observed in Section 1.3 this distinction between ontic and epistemic states is well-defined only if there is some assumption on what can in principle be known, represented here by the STH.

If we ask what is the evidence supporting the STH, the problem emerges that any measurement of the position and momentum of a classical body is afflicted with an uncertainty. What we really know from any carefully done experiment are expectation values Q and P and variances ΔQ and ΔP of positions and momenta. In fact, for a macroscopic body, we always have

$$2\Delta Q\Delta P \gg \hbar ,$$

where “ \gg ” represents many orders of magnitude (for the definition of variance, see e.g. [1], p. 223).

There can be different attitudes concerning this fact.

1. With improving techniques, the expression on the left-hand side will decrease eventually approaching zero. This is wrong if quantum mechanics is valid.
2. A sharp trajectory is just a model of a real motion. Thus, it is approximative and describes only some aspects of the motion. The model is considered as valid if the sharp trajectory lies within the “tube” in the phase space that is defined by the measured expectation values and variances. This corresponds well with the common experimental practice, as well as with Constructive Realism.
3. Another model of a real motion is a time-dependent probability distribution on the phase space that have suitable expectation values and variances. This seems to be a more accurate model of what is really observed.

If we assume that quantum mechanics is true then the sharp trajectories used in Points 2 and 3 do not exist in real world and are only a simplifying assumptions

allowing constructions of nice models. Actually, the probability distribution of Point 3 is not measurable because the pointlike states do not exist. It cannot therefore be considered as a purely epistemic state because no knowledge of the sharp trajectories is possible.

These considerations motivate an understanding of Newtonian mechanics that is different from the common one. Such an understanding is not new: the point of view that the statistical character of classical observational results must not only be due to inaccuracy of observational methods but also to genuine uncertainty of quantum origin is due to Exner [14], (“physical laws are only average laws”) p. 669, and Born [5] (the title: “Is classical mechanics really deterministic?”). It can be formulated as follows:

Exner-Born Conjecture *States of Newtonian systems that are described by sharp points of the phase space do not exist. Newtonian models that can approach the reality better are non-trivial probability distribution function on the phase space.*

The Conjecture suggests a change of interpretation of classical theories and, with it, a change of expectation of what is to be approximately obtained from quantum mechanics in the classical limit.

However, most physicists take the existence of sharp trajectories seriously and try to obtain them from quantum mechanics as exactly as possible. Hence, they focus at quantum states the phase-space picture of which is as sharp as possible. That are states with minimum uncertainty allowed by quantum mechanics. For one degree of freedom, described by coordinate \mathbf{q} and momentum \mathbf{p} , the uncertainty is given by the quantity

$$\nu = \frac{2\Delta\mathbf{q}\Delta\mathbf{p}}{\hbar} . \quad (2.1)$$

The states with minimum uncertainty $\nu = 1$ are, however, very special extremal states. Such states do exist for macroscopic quantum systems but are very difficult to prepare unlike the usual states of macroscopic systems that we observe around us. As explained at the beginning of the chapter, they also have properties that are strange from the point of view of classical theories and they are therefore not what can be successfully used for modelling of classical systems. Thus, there are some reasons to abandon STH. The Exner-Born interpretation is not only more realistic but it also makes TH 2.2 applicable to Newtonian mechanics.

Let us discuss the objectivity of fuzzy states of classical objects. In quantum mechanics, the basis of objectivity of dynamical properties is the objectivity of the conditions that define preparation procedures. In other words, if a property is uniquely determined by a preparation, then it is an objective property. If we look closely, one hindrance to try the same idea in classical theories is the custom to

speak always about initial data instead of preparations. An initial datum can be and mostly is a sharp state. The question of exactly how a sharp state can come into being is ignored. This in turn seems justified by the hypothesis that sharp states are objective, that is, they just exist by themselves. It seems however also possible to accept the idea that preparation procedures play the same basic role in the classical as in quantum physics. Then, the nature and form of necessary preparation procedures must be specified and the corresponding states described. Let us give an example.

Consider a gun in a position that is mechanically fixed and that shoots bullets using cartridges of a given provenance. All shots made under these conditions form an ensemble with expected trajectory $(\vec{Q}_{\text{gun}}(t), \vec{P}_{\text{gun}}(t))$ and the trajectory variance $(\Delta\vec{Q}_{\text{gun}}(t), \Delta\vec{P}_{\text{gun}}(t))$ that describe objective properties of the ensemble. The Newtonian model of this ensemble is the evolution $\rho_{\text{gun}}(\vec{Q}, \vec{P}; t)$ of a suitable distribution function on the phase space.

The simplest construction of a fuzzy model is to fix initial expectation values and variances of coordinates and momenta, $Q^k, \Delta Q^k, P^k, \Delta P^k$, and consider everything else as unknown. This opens the problem to application of Bayesian methods, see, e.g., [40], which recommend maximising entropy in the cases of missing knowledge (see Section E.2 and [40], Chapter 11). Let us define a fuzzy state called *maximum-entropy packet* (ME packet) as the phase-space distribution maximising entropy for given expectation values and variances of mechanical state coordinates. Of course, this is a particular choice that represents only one in a large number of possibilities.

For example, the above definition depends on the coordinates q and p , and it can be shown that it is not canonically invariant. Choose, e.g., q' and p' defined by the following canonical transformation:

$$\begin{aligned} q' &= \frac{1}{\sqrt{2}}q - \frac{1}{\sqrt{2}}p, \\ p' &= \frac{1}{\sqrt{2}}q + \frac{1}{\sqrt{2}}p. \end{aligned}$$

Then,

$$\begin{aligned} \Delta Q'^2 &= \left\langle \left[\frac{1}{\sqrt{2}}q - \frac{1}{\sqrt{2}}p - \frac{1}{\sqrt{2}}Q + \frac{1}{\sqrt{2}}P \right]^2 \right\rangle \\ &= \left\langle \frac{1}{2}(q - Q)^2 + \frac{1}{2}(p - P)^2 - (q - Q)(p - P) \right\rangle = \frac{1}{2}\Delta Q^2 + \frac{1}{2}\Delta P^2 - \langle (q - Q)(p - P) \rangle, \end{aligned}$$

where $\langle x \rangle$ represents the expectation value of quantity x in the state that is under consideration.

But $\langle (q - Q)(p - P) \rangle$ is the correlation function of the variables q and p and it is not determined by $Q, P, \Delta Q$ and ΔP . Hence, the condition that $Q, P, \Delta Q$ and

ΔP are fixed is in general not equivalent to Q' , P' , $\Delta Q'$ and $\Delta P'$ being fixed. One consequence of the definition of an ME packet not being canonically invariant is that the property of maximum entropy is not preserved by the dynamical evolution of the state.

The fact that our definition of the maximum entropy packet depends on the coordinates chosen for the description of the system and on the time instant when it is applied is not necessarily a serious hindrance for our project: we are just going to construct a model of a sufficiently fuzzy state. It is clear that such construction is inherently arbitrary. Any such model will do and it may be even advantageous to have some freedom. It seems plausible that relevant properties of the fuzzy states are independent of the details of their definition in some reasonable extent. Some suitable formulation of such assumptions must yet be found and their validity must be studied.

2.3 Classical ME packets

Let us first consider the sharp trajectories of a classical mechanical system S_c and their approximation by a quantum system S_q . For any comparison of Newtonian and quantum mechanics, it is necessary that the Newtonian canonical coordinates are chosen in such a way that there are reasonable quantum observables corresponding to them. For example, we ought to assume that the space coordinates are Cartesian. Let S_c have just one degree of freedom, canonical coordinates q and p and Hamiltonian

$$H = \frac{p^2}{2\mu} + V(q) , \quad (2.2)$$

where μ is a mass and $V(q)$ a potential function. The classical equations of motion are

$$\dot{q} = \frac{p}{\mu} , \quad \dot{p} = -\frac{dV}{dq} . \quad (2.3)$$

Their solution is a sharp trajectory,

$$q = q(t) , \quad p = p(t) ,$$

for every initial values $q(0)$ and $p(0)$.

Let us choose the corresponding quantum model S_q to be a system of one degree of freedom with position operator \mathbf{q} , momentum operator \mathbf{p} and spin 0. Operators \mathbf{q} and \mathbf{p} are related to the chosen coordinates q and p , e.g., through their spectra. Let the Hamiltonian be

$$H = \frac{\mathbf{p}^2}{2\mu} + V(\mathbf{q}) . \quad (2.4)$$

The Heisenberg equations of motion are

$$\dot{\mathbf{q}} = \frac{\mathbf{p}}{\mu} , \quad \dot{\mathbf{p}} = -\frac{dV}{d\mathbf{q}} . \quad (2.5)$$

Then the time dependence of position and momentum expectation values $Q = \langle \mathbf{q} \rangle$ and $P = \langle \mathbf{p} \rangle$ in a state $|\psi\rangle$ is

$$\dot{Q} = \frac{P}{\mu} , \quad \dot{P} = -\left\langle \frac{dV}{d\mathbf{q}} \right\rangle .$$

To evaluate the right-hand side of the second equation, let us expand the potential function in powers of $\mathbf{q} - Q$:

$$V(\mathbf{q}) = V(Q) + (\mathbf{q} - Q)\frac{dV}{dQ} + \frac{1}{2}(\mathbf{q} - Q)^2\frac{d^2V}{dQ^2} + \dots$$

so that

$$\frac{dV}{d\mathbf{q}} = \frac{dV}{dQ} + (\mathbf{q} - Q)\frac{d^2V}{dQ^2} + \frac{1}{2}(\mathbf{q} - Q)^2\frac{d^3V}{dQ^3} + \dots$$

If we take the expectation value of the last equation and use relations $\langle (\mathbf{q} - Q) \rangle = 0$ and $\langle (\mathbf{q} - Q)^2 \rangle = \Delta Q^2$, where ΔQ is the variance of \mathbf{q} in state $|\psi\rangle$, we obtain

$$\left\langle \frac{dV}{d\mathbf{q}} \right\rangle = \frac{dV}{dQ} + \frac{1}{2}\Delta Q^2\frac{d^3V}{dQ^3} + \dots$$

Let us assume that coordinate q and momentum p of S_c are obtained from the quantum model by formulas

$$q = Q , \quad p = P .$$

(This assumption is only natural if we worked with fuzzy classical models or if we are going to compare a fuzzy model with a sharp trajectory one.) Then, already for potentials of the third order, the quantum equations of motion for expectation values deviate from classical equation of motion for sharp trajectories. The quantum correction is proportional to the variation ΔQ . There are two important observations about this quantum correction. First, the correction is not proportional to \hbar . Second, the correction would be negligible for small ΔQ . Hence, the difference to the classical trajectory is smaller if the spread of the wave packet $|\psi\rangle$ over the space is smaller. This implies that the minimum-uncertainty wave packets give the best approximation to classical sharp trajectories.

However, for small ΔQ the variance ΔP is large, and ΔQ will quickly increase with time. Moreover, as already discussed, the minimum uncertainty packets have some further disadvantages.

Next, we turn to that maximum-entropy packets. Let us start the theory of such packets with the above system S_c of one degree of freedom and then generalise it

to any number of degrees. A fuzzy state is a distribution function $\rho(q, p)$ on the phase space spanned by Cartesian coordinates q and p . The function $\rho(q, p)$ is dimensionless and normalized by

$$\int \frac{dq dp}{v} \rho = 1 ,$$

where v is an auxiliary phase-space volume to make ρ dimensionless. The entropy of $\rho(q, p)$ can be defined by

$$\Sigma := - \int \frac{dq dp}{v} \rho \ln \rho .$$

The value of entropy will depend on v but most other results will not. Classical mechanics does not offer any idea of how to fix the value of v . We shall get a hint from quantum mechanics.

If we have chosen a different set of canonical coordinates, q' and p' , say, then the transformation

$$q = q(q', p') , \quad p = p(q', p') ,$$

being canonical, satisfies

$$\frac{\partial(q, p)}{\partial(q', p')} = 1 ,$$

where the left-hand side is the Jacobian of the transformation (see, e.g., [46], Section 46, p. 146). For the transformed distribution function,

$$\rho'(q', p') = \rho(q(q', p'), p(q', p')) ,$$

then holds that

$$\int \frac{dq dp}{v} \rho = \int \frac{dq' dp'}{v} \rho' , \quad \int \frac{dq dp}{v} \rho \ln \rho = \int \frac{dq' dp'}{v} \rho' \ln \rho' .$$

Hence, the normalisation condition and the value of entropy are invariant with respect to a general canonical transformation in Newtonian mechanics.

Let us now give a rigorous definition of the classical ME packets following [25, 32].

Definition 2.1 *ME packet is the distribution function $\rho[Q, P, \Delta Q, \Delta P]$ that maximizes the entropy subject to the conditions:*

$$\langle q \rangle = Q , \quad \langle q^2 \rangle = \Delta Q^2 + Q^2 , \tag{2.6}$$

and

$$\langle p \rangle = P , \quad \langle p^2 \rangle = \Delta P^2 + P^2 , \tag{2.7}$$

where the values of Q , P , ΔQ and ΔP are given.

We have used the abbreviation

$$\langle x \rangle = \int \frac{dq dp}{v} x \rho$$

for any function $x(q, p)$.

The explicit form of ME packets can be found using the partition-function method as it is derived, e.g., in Ref. [40], Chapter 11. The variational principle,

$$\delta \int \frac{dq dp}{v} (\rho \ln \rho + \lambda_0 \rho + \lambda_1 \rho q + \lambda_2 \rho p + \lambda_3 \rho q^2 + \lambda_4 \rho p^2) = 0 ,$$

where $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ and λ_4 are the five Lagrange multipliers corresponding to the normalisation condition and to the four conditions (2.6) and (2.7), yields

$$\rho = \frac{1}{Z(\lambda_1, \lambda_2, \lambda_3, \lambda_4)} \exp(-\lambda_1 q - \lambda_2 p - \lambda_3 q^2 - \lambda_4 p^2) , \quad (2.8)$$

so that the normalisation condition for ρ gives $\exp(1 + \lambda_0) = Z$, where

$$Z = \int \frac{dq dp}{v} \exp(-\lambda_1 q - \lambda_2 p - \lambda_3 q^2 - \lambda_4 p^2) \quad (2.9)$$

is the *partition function*. The integral is easy to calculate:

$$Z = \frac{\pi}{v \sqrt{\lambda_3 \lambda_4}} \exp \left(\frac{\lambda_1^2}{4\lambda_3} + \frac{\lambda_2^2}{4\lambda_4} \right) . \quad (2.10)$$

From the definition (2.9) of partition function, it follows that the expectation value of any monomial of the form $q^k p^l q^{2m} p^{2n}$ can be calculated with the help of partition-function method as follows:

$$\langle q^k p^l q^{2m} p^{2n} \rangle = \frac{(-1)^{\mathbf{N}}}{Z} \frac{\partial^{\mathbf{N}} Z}{\partial \lambda_1^k \partial \lambda_2^l \partial \lambda_3^m \partial \lambda_4^n} , \quad (2.11)$$

where $\mathbf{N} = k + l + 2m + 2n$ and Z is given by Eq. (2.10).

Observe that this allows to calculate the expectation value of a monomial in several different ways. Each of these ways, however, leads to the same result due the identities

$$\frac{\partial^2 Z}{\partial \lambda_1^2} = -\frac{\partial Z}{\partial \lambda_3} , \quad \frac{\partial^2 Z}{\partial \lambda_2^2} = -\frac{\partial Z}{\partial \lambda_4} ,$$

which are satisfied by the partition function.

In particular, the expressions for $\lambda_1, \lambda_2, \lambda_3$ and λ_4 in terms of $Q, P, \Delta Q$ and ΔP can be obtained by solving the equations

$$\frac{\partial \ln Z}{\partial \lambda_1} = -Q , \quad \frac{\partial \ln Z}{\partial \lambda_3} = -\Delta Q^2 - Q^2 ,$$

and

$$\frac{\partial \ln Z}{\partial \lambda_2} = -P, \quad \frac{\partial \ln Z}{\partial \lambda_4} = -\Delta P^2 - P^2.$$

The result is:

$$\lambda_1 = -\frac{Q}{\Delta Q^2}, \quad \lambda_3 = \frac{1}{2\Delta Q^2}, \quad (2.12)$$

and

$$\lambda_2 = -\frac{P}{\Delta P^2}, \quad \lambda_4 = \frac{1}{2\Delta P^2}. \quad (2.13)$$

Substituting Eqs. (2.12) and (2.13) into Eq. (2.8), we obtain the distribution function of a one-dimensional ME packet. The generalization to any number of dimensions is easy:

Theorem 2.1 *The distribution function of the ME packet for a system S_c of n degrees of freedom with given expectation values and variances $Q_1, \dots, Q_n, \Delta Q_1, \dots, \Delta Q_n$ of coordinates and $P_1, \dots, P_n, \Delta P_1, \dots, \Delta P_n$ of momenta, is*

$$\begin{aligned} \rho[Q, P, \Delta Q, \Delta P](q, p) \\ = \left(\frac{v}{2\pi}\right)^n \prod_{k=1}^n \left(\frac{1}{\Delta Q_k \Delta P_k} \exp \left[-\frac{(q_k - Q_k)^2}{2\Delta Q_k^2} - \frac{(p_k - P_k)^2}{2\Delta P_k^2} \right] \right), \end{aligned} \quad (2.14)$$

where $Q, P, \Delta Q, \Delta P$ q and p stand for n -tuples of values, e.g., $Q \equiv Q_1, \dots, Q_n$, etc.

Formula (2.14) holds for general canonical coordinates, not only for the Cartesian ones.

The state of system S_c is described by distribution function $\rho[Q, P, \Delta Q, \Delta P](q, p)$ which is determined by $4n$ values. In this way, to describe the mechanical degrees of freedom, we need twice as many variables as the standard mechanics. The doubling of state coordinates is due to the necessity to define a fuzzy distribution rather than a sharp trajectory.

We observe that all expectation values obtained from ρ are independent of v and that the right-hand side of equation (2.14) is a Gaussian distribution in agreement with Jaynes' conjecture that the maximum entropy principle gives the Gaussian distribution if the only conditions are fixed values of the first two moments.

As ΔQ and ΔP approach zero, ρ becomes a δ -function and the state becomes sharp. For some quantities this limit is sensible for others it is not. In particular, the entropy, which can easily be calculated,

$$\Sigma = 1 + \ln \frac{2\pi \Delta Q \Delta P}{v},$$

diverges to $-\infty$. This is due to a general difficulty in giving a definition of entropy for a continuous system that would be satisfactory in every respect (see [40], Section

12.3). What one could do is to divide the phase space into cells of volume v so that $\Delta Q \Delta P$ could not be chosen smaller than v . Then, the limit $\Delta Q \Delta P \rightarrow v$ of entropy would make more sense.

The importance of the ME packets for our theory is expressed by:

ME-Packet Conjecture *For most mechanical objects \mathcal{S} , all measurable predictions of Newtonian mechanics can be obtained from a classical model S_c described by Theorem 2.1.*

This can be considered as a more specific form of Exner-Born Conjecture.

2.4 Polynomial potential function

Here, an account of some aspects of Newtonian dynamics of ME packets is given with the aim to compare them with, or obtain them in some approximation from, the dynamics of corresponding quantum systems later. This comparison is easy for polynomial potential functions.

Let the Hamiltonian of S_c has the form (2.2) so that the equations of motion are (2.3). The general solution to these equations can be written in the form

$$q(t) = \bar{q}(t; q, p) , \quad p(t) = \bar{p}(t; q, p) , \quad (2.15)$$

where

$$\bar{q}(0; q, p) = q , \quad \bar{p}(0; q, p) = p , \quad (2.16)$$

q and p being arbitrary initial values. This implies for the time dependence of the expectation values and variances, if the initial state is an ME packet:

$$\bar{Q}(t) = \langle \bar{q}(t; q, p) \rangle , \quad \Delta \bar{Q}(t) = \sqrt{\langle \bar{q}^2(t; q, p) \rangle - \langle \bar{q}(t; q, p) \rangle^2} \quad (2.17)$$

and

$$\bar{P}(t) = \langle \bar{p}(t; q, p) \rangle , \quad \Delta \bar{P}(t) = \sqrt{\langle \bar{p}^2(t; q, p) \rangle - \langle \bar{p}(t; q, p) \rangle^2} . \quad (2.18)$$

We introduce the notation $\bar{Q}(t)$, $\bar{P}(t)$, $\Delta \bar{Q}(t)$ and $\Delta \bar{P}(t)$ to distinguish the expectation values and variances of time-dependent coordinates and momenta from the values Q , P , ΔQ and ΔP that define the EM packet.

Let us first consider the special case of at most quadratic potential:

$$V(q) = V_0 + V_1 q + \frac{1}{2} V_2 q^2 , \quad (2.19)$$

where V_k are constants with suitable dimensions. If $V_1 = V_2 = 0$, we have a free particle, if $V_2 = 0$, it is a particle in a homogeneous force field and if $V_2 \neq 0$, it is an harmonic or anti-harmonic oscillator.

For potential (2.19), the dynamical equations are linear and their general solution (2.15) has the form

$$\bar{q}(t) = f_0(t) + qf_1(t) + pf_2(t) , \quad (2.20)$$

$$\bar{p}(t) = g_0(t) + qg_1(t) + pg_2(t) , \quad (2.21)$$

where $f_0(0) = f_2(0) = g_0(0) = g_1(0) = 0$ and $f_1(0) = g_2(0) = 1$. If $V_2 \neq 0$, the functions are

$$f_0(t) = -\frac{V_1}{V_2}(1 - \cos \omega t) , \quad f_1(t) = \cos \omega t , \quad f_2(t) = \frac{1}{\xi} \sin \omega t , \quad (2.22)$$

$$g_0(t) = -\xi \frac{V_1}{V_2} \sin \omega t , \quad g_1(t) = -\xi \sin \omega t , \quad g_2(t) = \cos \omega t , \quad (2.23)$$

where

$$\xi = \sqrt{\mu V_2} , \quad \omega = \sqrt{\frac{V_2}{\mu}} .$$

Only for $V_2 > 0$, the functions remain bounded. If $V_2 = 0$, we obtain

$$f_0(t) = -\frac{V_1}{2\mu} t^2 , \quad f_1(t) = 1 , \quad f_2(t) = \frac{t}{\mu} , \quad (2.24)$$

$$g_0(t) = -V_1 t , \quad g_1(t) = 0 , \quad g_2(t) = 1 . \quad (2.25)$$

The time dependence of expectation values and variances resulting from Eqs. (2.15), (2.6) and (2.7) are

$$\bar{Q}(t) = f_0(t) + Qf_1(t) + Pf_2(t) \quad (2.26)$$

and

$$\begin{aligned} \Delta \bar{Q}^2(t) + \bar{Q}^2(t) &= f_0^2(t) + (\Delta Q^2 + Q^2)f_1^2(t) + (\Delta P^2 + P^2)f_2^2(t) \\ &\quad + 2Qf_0(t)f_1(t) + 2Pf_0(t)f_2(t) + 2\langle qp \rangle f_1(t)f_2(t) . \end{aligned} \quad (2.27)$$

For the last term, we have from Eq. (2.11)

$$\langle qp \rangle = \frac{1}{Z} \frac{\partial^2 Z}{\partial \lambda_1 \partial \lambda_2} .$$

Using Eqs. (2.10), (2.12) and (2.13), we obtain from Eq. (2.27)

$$\Delta \bar{Q}(t) = \sqrt{f_1^2(t) \Delta Q^2 + f_2^2(t) \Delta P^2} . \quad (2.28)$$

Similarly,

$$\bar{P}(t) = g_0(t) + Qg_1(t) + Pg_2(t) , \quad (2.29)$$

$$\Delta \bar{P}(t) = \sqrt{g_1^2(t) \Delta Q^2 + g_2^2(t) \Delta P^2} . \quad (2.30)$$

We observe: if functions $f_1(t)$, $f_2(t)$, $g_1(t)$ and $g_2(t)$ remain bounded, the variances also remain bounded and the predictions are possible in arbitrary long intervals of time. Otherwise, there will always be only limited time intervals in which the theory can make reasonable predictions. We can also see that the evolution $\bar{Q}(t)$ and $\bar{P}(t)$ coincides with the sharp trajectories (2.20) and (2.21). In particular, it is independent of ΔQ and ΔP , which is a well-known property of potential (2.19). In general, $\bar{Q}(t)$ and $\bar{P}(t)$ will depend not only on initial Q and P , but also on ΔQ and ΔP .

From formulas (2.28) and (2.30) we can also see that the ME packet form is not preserved by the evolution (the entropy ceases to be maximal). First, both variances must increase near $t = 0$. Second, the entropy must stay constant because it is preserved by the dynamics. Third, the relation between entropy and ν is fixed for ME packets.

After these preparation remarks, we turn to a general polynomial potential of degree N ,

$$V(q) = \sum_{k=0}^N \frac{1}{k!} V_k q^k, \quad (2.31)$$

and study the general time derivatives of \bar{Q} , \bar{P} , $\Delta\bar{Q}$, $\Delta\bar{P}$. We shall need some results of this study for the proof of Theorem 3.2 on the classical limit. The right-hand sides of dynamical equation (2.3) for the potential function (2.31) are polynomials in \bar{q} and \bar{p} :

$$\frac{\partial \bar{q}}{\partial t} = \frac{1}{\mu} \bar{p}, \quad \frac{\partial \bar{p}}{\partial t} = - \sum_{k=0}^{N-1} \frac{1}{k!} V_{k+1} \bar{q}^k.$$

In general, for the K -th time derivatives, we obtain

$$\begin{aligned} \frac{\partial^K \bar{q}}{\partial t^K} &= A_K(\bar{q}, \bar{p}), \\ \frac{\partial^K \bar{p}}{\partial t^K} &= B_K(\bar{q}, \bar{p}), \end{aligned}$$

where A_K and B_K are polynomials in \bar{q} and \bar{p} . The proof is by mathematical induction: For the first derivative, the claim is true. If it is true for the K -th derivative, then we have for the $K + 1$ -th one:

$$\begin{aligned} \frac{\partial^{K+1} \bar{q}}{\partial t^{K+1}} &= \frac{\bar{p}}{\mu} \frac{\partial A_K(\bar{q}, \bar{p})}{\partial \bar{q}} - \sum_{k=0}^{N-1} \frac{1}{k!} V_{k+1} \bar{q}^k \frac{\partial A_K(\bar{q}, \bar{p})}{\partial \bar{p}}, \\ \frac{\partial^{K+1} \bar{p}}{\partial t^{K+1}} &= \frac{\bar{p}}{\mu} \frac{\partial B_K(\bar{q}, \bar{p})}{\partial \bar{q}} - \sum_{k=0}^{N-1} \frac{1}{k!} V_{k+1} \bar{q}^k \frac{\partial B_K(\bar{q}, \bar{p})}{\partial \bar{p}}, \end{aligned}$$

which are again polynomials in \bar{q} and \bar{p} . The equations for the time derivatives of expectation values are

$$\left(\frac{d^K \bar{Q}}{dt^K}\right)_0 = \langle A_K(q, p) \rangle, \quad \left(\frac{d^K \bar{P}}{dt^K}\right)_0 = \langle B_K(q, p) \rangle,$$

and the calculation of all expectation values can be reduced to that of $q^k p^l$ -products.

Slightly more complicated equations hold for the time derivatives of variances. For the first derivatives, we obtain

$$\frac{d\Delta \bar{Q}(t)}{dt} = \frac{1}{\mu \Delta \bar{Q}(t)} (\langle \bar{q} \bar{p} \rangle - \langle \bar{q} \rangle \langle \bar{p} \rangle), \quad (2.32)$$

$$\frac{d\Delta \bar{P}(t)}{dt} = -\frac{1}{\Delta \bar{P}(t)} \left(\left\langle \bar{p} \sum_{k=0}^{N-1} \frac{1}{k!} V_{k+1} \bar{q}^k \right\rangle - \langle \bar{p} \rangle \left\langle \sum_{k=0}^{N-1} \frac{1}{k!} V_{k+1} \bar{q}^k \right\rangle \right). \quad (2.33)$$

To get any further, we need the following property:

$$\langle q^k p^l \rangle = \langle q^k \rangle \langle p^l \rangle.$$

Indeed, we obtain easily from Eq. (2.11) that

$$\begin{aligned} \langle q^k p^l \rangle &= \left[(-1)^k \exp\left(-\frac{\lambda_1^2}{4\lambda_3}\right) \frac{\partial^k}{\partial \lambda_1^k} \exp\left(\frac{\lambda_1^2}{4\lambda_3}\right) \right] \\ &\quad \times \left[(-1)^l \exp\left(-\frac{\lambda_2^2}{4\lambda_4}\right) \frac{\partial^l}{\partial \lambda_2^l} \exp\left(\frac{\lambda_2^2}{4\lambda_4}\right) \right]. \end{aligned} \quad (2.34)$$

From Eqs. (2.32) and (2.33), it then follows immediately that

$$\left(\frac{d\Delta \bar{Q}}{dt}\right)_0 = 0, \quad \left(\frac{d\Delta \bar{P}}{dt}\right)_0 = 0,$$

This implies that, in calculating higher time derivatives of the right-hand sides of Eqs. (2.32) and (2.33), we can ignore the variances in the denominator, so that we obtain

$$\left(\frac{d^K \Delta \bar{Q}}{dt^K}\right)_0 = \frac{1}{\mu \Delta \bar{Q}} \left[\frac{\partial^{K-1}}{\partial t^{K-1}} (\langle \bar{q} \bar{p} \rangle - \langle \bar{q} \rangle \langle \bar{p} \rangle) \right]_0, \quad (2.35)$$

$$\begin{aligned} &\left(\frac{d^K \Delta \bar{P}}{dt^K}\right)_0 \\ &= -\frac{1}{\Delta \bar{P}} \left[\frac{\partial^{K-1}}{\partial t^{K-1}} \left(\left\langle \bar{p} \sum_{k=0}^{N-1} \frac{1}{k!} V_{k+1} \bar{q}^k \right\rangle - \langle \bar{p} \rangle \left\langle \sum_{k=0}^{N-1} \frac{1}{k!} V_{k+1} \bar{q}^k \right\rangle \right) \right]_0. \end{aligned} \quad (2.36)$$

Now, it is easy to show that the right-hand sides will be expressions constructed from expectation values of polynomials in q and p by the same argument as that used for expectation values.

It follows that all time derivatives of $Q(t)$, $P(t)$, $\Delta Q(t)$ and $\Delta P(t)$ at $t = 0$ can be calculated by an iterative application of Eqs. (2.3) and then using Formula (2.11) (in [25], the first four time derivatives for a fourth-degree potential function have been calculated).

Finally, we have the following Lemma.

Lemma 2.1 *The expectation value $\langle q^x p^y \rangle_c$ for any non-negative integers x and y is a polynomial,*

$$\langle q^x p^y \rangle_{cl} = X(Q, P, \Delta Q, \Delta P) ,$$

with integer coefficients. The term of the highest order of ΔQ and ΔP in X has the form,

$$A_m B_n Q^{x-2m} P^{y-2n} \Delta Q^{2m} \Delta P^{2n} ,$$

where A_m and B_n are positive integers,

$$m = \left[\frac{x}{2} \right] , \quad n = \left[\frac{y}{2} \right]$$

and $[a]$ is the integer part of a real number a , i.e., the largest integer not larger than a .

Proof Consider first the case of $x = 0, 1$ so that $m = 0$, and similarly for y and n . An easy calculation using Eq. (2.34) gives:

$$A_0 = 1 , \quad B_0 = 1 .$$

For $m \neq 0$ and $n \neq 0$, Eq. (2.34) implies that X is a product of two functions,

$$X(Q, P, \Delta Q, \Delta P) = Y_x(\lambda_1, \lambda_3) Y_y(\lambda_2, \lambda_4) ,$$

where Eqs. (2.12) and (2.13) must be substituted for $\lambda_1, \lambda_2, \lambda_3$ and λ_4 .

Let us show that

$$\frac{\partial^{2m-1}}{\partial \lambda_1^{2m-1}} \exp \left(\frac{\lambda_1^2}{4\lambda_3} \right) = \left[\sum_{k=1}^m b_{m,k} \frac{\lambda_1^{2k-1}}{(2\lambda_3)^{m+k-1}} \right] \exp \left(\frac{\lambda_1^2}{2\lambda_3} \right) , \quad (2.37)$$

$$\frac{\partial^{2m}}{\partial \lambda_1^{2m}} \exp \left(\frac{\lambda_1^2}{4\lambda_3} \right) = \left[\sum_{k=1}^{m+1} a_{m,k} \frac{\lambda_1^{2k-2}}{(2\lambda_3)^{m+k-1}} \right] \exp \left(\frac{\lambda_1^2}{4\lambda_3} \right) , \quad (2.38)$$

where $b_{m,k}$ and $a_{m,k}$ are positive integer coefficients. It then follows that Y_x is a polynomial of Q and ΔQ^2 and Y_y that of P and ΔP^2 , because

$$\frac{\lambda_1^{2k-1}}{(2\lambda_3)^{m+k-1}} = \frac{\lambda_1^{2k-1}}{(2\lambda_3)^{2k-1}} \frac{1}{(2\lambda_3)^{m-k}} = -Q^{2k-1} (\Delta Q^2)^{m-k} ,$$

and similarly for Y_y .

The proof of Eqs. (2.37) and (2.38) by mathematical induction consists of the following steps. First, we easily obtain:

$$\begin{aligned}\frac{\partial}{\partial \lambda_1} \exp\left(\frac{\lambda_1^2}{4\lambda_3}\right) &= \frac{\lambda_1}{2\lambda_3} \exp\left(\frac{\lambda_1^2}{4\lambda_3}\right) , \\ \frac{\partial^2}{\partial \lambda_1^2} \exp\left(\frac{\lambda_1^2}{4\lambda_3}\right) &= \left[\frac{1}{2\lambda_3} + \left(\frac{\lambda_1}{2\lambda_3}\right)^2\right] \exp\left(\frac{\lambda_1^2}{4\lambda_3}\right) ,\end{aligned}$$

which coincide with Eqs. (2.37) and (2.38) for $m = 1$.

Second, assuming the validity of Eq. (2.37), we calculate the even derivative from the odd one:

$$\frac{\partial^{2m}}{\partial \lambda_1^{2m}} \exp\left(\frac{\lambda_1^2}{4\lambda_3}\right) = \frac{\partial}{\partial \lambda_1} \left[\left(\sum_{k=1}^m b_{m,k} \frac{\lambda_1^{2k-1}}{(2\lambda_3)^{m+k-1}} \right) \exp\left(\frac{\lambda_1^2}{4\lambda_3}\right) \right] ,$$

which, after a simple rearrangement, becomes (2.38) with

$$\begin{aligned}a_{m,1} &= b_{m,1} , \\ a_{m,k} &= b_{m,k-1} + (2k-1)b_{m,k}\end{aligned}$$

for $k = 2, \dots, m$, and

$$a_{m,m+1} = b_{m,m} .$$

Similarly,

$$\frac{\partial^{2m+1}}{\partial \lambda_1^{2m+1}} \exp\left(\frac{\lambda_1^2}{4\lambda_3}\right) = \frac{\partial}{\partial \lambda_1} \left[\left(\sum_{k=1}^{m+1} a_{m,k} \frac{\lambda_1^{2k-2}}{(2\lambda_3)^{m+k-1}} \right) \exp\left(\frac{\lambda_1^2}{4\lambda_3}\right) \right] ,$$

which becomes (2.37), if m is replaced by $m+1$, with

$$b_{m+1,k} = a_{m,k} + 2ka_{m,k+1}$$

for $k = 1, \dots, m$, and

$$b_{m+1,m+1} = a_{m,m+1} .$$

Finally, substituting from Eq. (2.12) for λ_1 and λ_3 into Eqs. (2.37) and (2.38), we obtain:

$$\frac{\lambda_1^{2k-1}}{(2\lambda_3)^{m+k-1}} = -Q^{2k-1} \Delta Q^{2m-1k} ,$$

and

$$\frac{\lambda_1^{2k-2}}{(2\lambda_3)^{m+k-1}} = -Q^{2k-2} \Delta Q^{2m-1k+2} .$$

In particular, the terms of the highest order in ΔQ are (after the multiplication by $(-1)^{x+y}$)

$$b_{m,1}Q\Delta Q^{2m-2}$$

for $x = 2m - 1$, and

$$a_{m,1}\Delta Q^{2m}$$

for $x = 2m$.

The whole procedure can be repeated in the same form for P and ΔP thus showing Eqs. (2.37) and (2.38) with λ_1 and λ_3 replaced by λ_2 and λ_4 . Now, the claim of the Lemma follows with $A_m = a_{m,1}$ and $B_m = b_{m,1}$ for all positive integers m and n , **QED**.

Chapter 3

Classical limit

Here, we first define and study quantum ME packets and then compare their dynamical trajectories with those of their classical counterparts for polynomial potential functions. The main result is a theorem, on which our new notion of classical limit is based.

3.1 Quantum ME packets

According to TH 2.1, the conditions defining the classical model determine the preparation of the corresponding quantum model. The foregoing chapter introduced ME packets as models of mechanical systems. The corresponding quantum models are defined by:

Definition 3.1 *Let the quantum model S_q of object \mathcal{S} has spin 0, position \mathbf{q} and momentum \mathbf{p} . State T that maximizes von Neumann entropy under the conditions*

$$\text{tr}[\mathsf{T}\mathbf{q}] = Q, \quad \text{tr}[\mathsf{T}\mathbf{q}^2] = Q^2 + \Delta Q^2, \quad (3.1)$$

$$\text{tr}[\mathsf{T}\mathbf{p}] = P, \quad \text{tr}[\mathsf{T}\mathbf{p}^2] = P^2 + \Delta P^2, \quad (3.2)$$

where Q , P , ΔQ and ΔP are given numbers, is called quantum ME packet.

Then, TH 2.1 implies that the quantum ME packet for Q , P , ΔQ and ΔP is the quantum model S_q corresponding to the classical ME packet with the same values of Q , P , ΔQ and ΔP .

Von Neumann entropy Σ of a given state T (see, e.g., [58], Section 9-1) is

$$\Sigma(\mathsf{T}) = -\text{tr}[\mathsf{T} \ln(\mathsf{T})]. \quad (3.3)$$

As each T must have a discrete spectrum with positive eigenvalues t_k (see [63], p. 209), we have

$$\Sigma(\mathsf{T}) = -\sum_k t_k \ln(t_k).$$

To calculate state T , we pretend that all s.a. operators that occur in the calculation are just $n \times n$ hermitian matrices, as it is common in quantum mechanics courses. Real proofs are more difficult, but we assume that they can be given. Then, we use the method of Lagrange multipliers as in the classical case. The variational principle for the maximum entropy yields the following equation:

$$d\Sigma - \lambda_0 d \operatorname{tr}(\mathsf{T}) - \lambda_1 d \operatorname{tr}(\mathsf{T}\mathbf{q}) - \lambda_2 d \operatorname{tr}(\mathsf{T}\mathbf{p}) - \lambda_3 d \operatorname{tr}(\mathsf{T}\mathbf{q}^2) - \lambda_4 d \operatorname{tr}(\mathsf{T}\mathbf{p}^2) = 0 . \quad (3.4)$$

Choosing an orthonormal basis $\{|n\rangle\}$ of the Hilbert space \mathbf{H} of \mathcal{S} , the differentials of the terms that are linear in T can be brought to the form:

$$d \operatorname{tr}[\mathsf{T}\mathbf{x}] = \sum_{mn} x_{nm} dT_{mn}$$

for any observable \mathbf{x} . Although not all elements of the matrix dT_{mn} are independent (it is a hermitian matrix), we can proceed as if they were because the matrix x_{nm} must also be hermitian. The only problem is to calculate $d\Sigma$. We have the following

Lemma 3.1

$$d\Sigma = - \sum_{mn} [\delta_{mn} + (\ln T)_{mn}] dT_{mn} . \quad (3.5)$$

Proof Let U be a unitary matrix that diagonalizes T ,

$$\mathsf{U}^\dagger \mathsf{T} \mathsf{U} = \mathsf{R} ,$$

where R is a diagonal matrix with elements R_n . Then $\Sigma = - \sum_n R_n \ln R_n$. Correction to R_n if T changes to $\mathsf{T} + d\mathsf{T}$ can be calculated by the first-order formula of the stationary perturbation theory (see, e.g., [1], p. 276). This theory is usually applied to Hamiltonians but it holds for any perturbed hermitian operator. Moreover, the formula is exact for infinitesimal perturbations. Thus,

$$R_n \mapsto R_n + \sum_{kl} U_{kn}^* U_{ln} dT_{kl} .$$

In this way, we obtain

$$\begin{aligned} d\Sigma &= - \sum_n \left(R_n + \sum_{kl} U_{kn}^* U_{ln} dT_{kl} \right) \\ &\quad \times \ln \left[R_n \left(1 + \frac{1}{R_n} \sum_{rs} U_{rn}^* U_{sn} dT_{rs} \right) \right] + \sum_n R_n \ln R_n \\ &= - \sum_n \left[\ln R_n \sum_{kl} U_{kn}^* U_{ln} dT_{kl} + \sum_{kl} U_{kn}^* U_{ln} dT_{kl} \right] \\ &= - \sum_{kl} [\delta_{kl} + (\ln \mathsf{T})_{kl}] dT_{kl} , \end{aligned}$$

QED.

With the help of Lemma 3.1, Eq. (3.4) becomes

$$tr \left((-1 - \ln T - \lambda_0 - \lambda_1 \mathbf{q} - \lambda_2 \mathbf{p} - \lambda_3 \mathbf{q}^2 - \lambda_4 \mathbf{p}^2) dT \right) = 0 ,$$

which must hold for any dT . Hence, we have

$$T = \exp(-\lambda_0 - 1 - \lambda_1 \mathbf{q} - \lambda_2 \mathbf{p} - \lambda_3 \mathbf{q}^2 - \lambda_4 \mathbf{p}^2) . \quad (3.6)$$

This can be written as

$$\exp(-\lambda_0 - 1) \exp(-\lambda_1 \mathbf{q} - \lambda_2 \mathbf{p} - \lambda_3 \mathbf{q}^2 - \lambda_4 \mathbf{p}^2)$$

$\exp(-\lambda_0 - 1)$ being just a number that can be considered as a normalisation constant. Taking trace of Eq. (3.6), we obtain

$$e^{-\lambda_0 - 1} = \frac{1}{Z(\lambda_1, \lambda_2, \lambda_3, \lambda_4)} ,$$

where Z is the partition function,

$$Z(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = tr[\exp(-\lambda_1 \mathbf{q} - \lambda_2 \mathbf{p} - \lambda_3 \mathbf{q}^2 - \lambda_4 \mathbf{p}^2)] . \quad (3.7)$$

Thus, the state operator has the form

$$T = \frac{1}{Z(\lambda_1, \lambda_2, \lambda_3, \lambda_4)} \exp(-\lambda_1 \mathbf{q} - \lambda_2 \mathbf{p} - \lambda_3 \mathbf{q}^2 - \lambda_4 \mathbf{p}^2) . \quad (3.8)$$

At this stage, the quantum theory begins to differ from the classical one. It turns out that, for the case of non-commuting operators in the exponent of the partition function, formula (2.11) is not valid in general. We can only show that it holds for the first derivatives.

Lemma 3.2 *Let A and B be Hermitian matrices. Then*

$$\frac{d^n}{d\lambda^n} \exp(A + B\lambda) = B^n \exp(A + B\lambda) \quad (3.9)$$

if A and B commute and

$$\frac{d}{d\lambda} tr[\exp(A + B\lambda)] = tr[B \exp(A + B\lambda)] \quad (3.10)$$

in general.

Proof Let us express the exponential function as a series:

$$\begin{aligned} d[\exp(\mathbf{A} + \mathbf{B}\lambda)] &= \sum_{n=0}^{\infty} \frac{1}{n!} [d(\mathbf{A} + \mathbf{B}\lambda)^n] \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left[\sum_{k=1}^n (\mathbf{A} + \mathbf{B}\lambda)^{k-1} \mathbf{B} (\mathbf{A} + \mathbf{B}\lambda)^{n-k} \right] d\lambda . \quad (3.11) \end{aligned}$$

For the first part of the Lemma, \mathbf{B} commutes with $\mathbf{A} + \mathbf{B}\lambda$ and can thus be brought to the left in each of the products and we obtain (3.9) by calculating higher derivatives by the same method.

For the second part, we take trace of both sides of Eq. (3.11) and use the invariance of trace of a product with respect to any cyclic permutation of the factors,

$$\begin{aligned} d \operatorname{tr}[\exp(\mathbf{A} + \mathbf{B}\lambda)] &= \sum_{n=0}^{\infty} \frac{1}{n!} \operatorname{tr}[d(\mathbf{A} + \mathbf{B}\lambda)^n] \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \operatorname{tr} \left[\sum_{k=1}^n (\mathbf{A} + \mathbf{B}\lambda)^{k-1} \mathbf{B} (\mathbf{A} + \mathbf{B}\lambda)^{n-k} \right] d\lambda \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=1}^n \operatorname{tr} [\mathbf{B} (\mathbf{A} + \mathbf{B}\lambda)^{n-1}] d\lambda = \operatorname{tr}[\mathbf{B} \exp(\mathbf{A} + \mathbf{B}\lambda)] d\lambda , \end{aligned}$$

which is Eq. (3.10), **QED**.

The proof of Lemma 3.2 shows why formula (2.11) is not valid for higher derivatives than the first in the quantum case: the operator B does not commute with $\mathbf{A} + \mathbf{B}\lambda$ and cannot be shifted from its position to the first position in product

$$(\mathbf{A} + \mathbf{B}\lambda)^k \mathbf{B} (\mathbf{A} + \mathbf{B}\lambda)^l .$$

Only for the first derivative, it can be brought there by a suitable cyclic permutation. However, each commutator $[\mathbf{B}, (\mathbf{A} + \mathbf{B}\lambda)]$ is proportional to \hbar . Hence, formula (2.11) with higher derivatives is the leading term in the expansion of expectation values in powers of \hbar .

Together with Eq. (3.7), Lemma 3.2 implies that:

$$\frac{\partial \ln Z}{\partial \lambda_1} = -Q , \quad \frac{\partial \ln Z}{\partial \lambda_3} = -Q^2 - \Delta Q^2 \quad (3.12)$$

and

$$\frac{\partial \ln Z}{\partial \lambda_2} = -P , \quad \frac{\partial \ln Z}{\partial \lambda_4} = -P^2 - \Delta P^2 . \quad (3.13)$$

The values of the multipliers can then be calculated from Eqs. (3.12) and (3.13), if the form of the partition function is known.

Variational methods can find locally extremal values that are not necessarily maxima. We can however prove that our state operator maximizes the entropy. The proof is based on the generalized Gibbs' inequality,

$$\text{tr}[\mathbf{T} \ln \mathbf{T} - \mathbf{T} \ln \mathbf{S}] \geq 0$$

for all pairs $\{\mathbf{T}, \mathbf{S}\}$ of state operators (for proof of the inequality, see [58], p. 264). The proof of maximality is then analogous to the “classical” proof (see, e.g., [40], p. 357). The first proof of maximality in the quantum case was given by von Neumann [55].

The state operator (3.8) can be inserted into formula (3.3) to give the value of the maximum entropy,

$$\Sigma = \ln Z + \lambda_1 \langle \mathbf{q} \rangle + \lambda_2 \langle \mathbf{p} \rangle + \lambda_3 \langle \mathbf{q}^2 \rangle + \lambda_4 \langle \mathbf{p}^2 \rangle . \quad (3.14)$$

This, together with Eqs. (3.12) and (3.13) can be considered as the Legendre transformation from the logarithm of partition function to the entropy,

$$\ln Z(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \mapsto \Sigma(\langle \mathbf{q} \rangle, \langle \mathbf{p} \rangle, \langle \mathbf{q}^2 \rangle, \langle \mathbf{p}^2 \rangle) .$$

One can observe that our modified Newtonian mechanics is quite similar to ordinary thermodynamics. For example, in thermodynamics, the physical meaning of the Lagrange multiplier λ in the term $\lambda \langle E \rangle$ of the variational principle is the inverse of some energy typical for the system in the state of maximal entropy, such as the expected energy of a molecule. In Newtonian mechanics, the meaning is that of the inverses of some typical expected values of coordinates, momenta and of their squares in the maximum entropy states.

3.2 Diagonal representation

The exponent in Eq. (3.8) can be written in the form

$$\frac{\lambda_1^2}{4\lambda_3} + \frac{\lambda_2^2}{4\lambda_4} - 2\sqrt{\lambda_3\lambda_4}\mathbf{K} , \quad (3.15)$$

where

$$\mathbf{K} = \frac{1}{2}\sqrt{\frac{\lambda_3}{\lambda_4}} \left(\mathbf{q} + \frac{\lambda_1}{2\lambda_3} \right)^2 + \frac{1}{2}\sqrt{\frac{\lambda_4}{\lambda_3}} \left(\mathbf{p} + \frac{\lambda_2}{2\lambda_4} \right)^2 \quad (3.16)$$

is an operator acting on the Hilbert space of our system. \mathbf{K} has the form of the Hamiltonian¹ of a harmonic oscillator with coordinate \mathbf{u} and momentum \mathbf{w}

$$\mathbf{u} = \mathbf{q} + \frac{\lambda_1}{2\lambda_3} , \quad \mathbf{w} = \mathbf{p} + \frac{\lambda_2}{2\lambda_4} , \quad (3.17)$$

¹The operator \mathbf{K} must not be confused with the Hamiltonian \mathbf{H} of our system, which can be arbitrary.

that satisfy the commutation relation $[u, w] = i\hbar$. The oscillator has mass $M = \sqrt{\lambda_3/\lambda_4}$ and frequency 1. The normalized eigenvectors $|k\rangle$ of the operator form a basis in the Hilbert space of our system defining what we shall call *diagonal representation*. The eigenvalues of K are $\hbar/2 + \hbar k$. As is usual in dealing with a harmonic oscillator, we introduce the “annihilation” operator A such that

$$AA^\dagger - A^\dagger A = 1, \quad (3.18)$$

$$u = \sqrt{\frac{\hbar}{2M}}(A + A^\dagger), \quad (3.19)$$

$$w = -i\sqrt{\frac{\hbar M}{2}}(A - A^\dagger), \quad (3.20)$$

$$K = \frac{\hbar}{2}(A^\dagger A + AA^\dagger), \quad (3.21)$$

$$A|k\rangle = \sqrt{k}|k-1\rangle, \quad (3.22)$$

$$A^\dagger|k\rangle = \sqrt{k+1}|k+1\rangle. \quad (3.23)$$

To calculate Z in the diagonal representation is easy:

$$\begin{aligned} Z &= \text{tr} \left[\exp \left(\frac{\lambda_1^2}{4\lambda_3} + \frac{\lambda_2^2}{4\lambda_4} - 2\sqrt{\lambda_3\lambda_4}K \right) \right] \\ &= \sum_{k=0}^{\infty} \langle k | \exp \left(\frac{\lambda_1^2}{4\lambda_3} + \frac{\lambda_2^2}{4\lambda_4} - 2\sqrt{\lambda_3\lambda_4}K \right) | k \rangle \\ &= \exp \left(\frac{\lambda_1^2}{4\lambda_3} + \frac{\lambda_2^2}{4\lambda_4} - \hbar\sqrt{\lambda_3\lambda_4} \right) \sum_{k=0}^{\infty} \exp(-2\hbar\sqrt{\lambda_3\lambda_4}k). \end{aligned}$$

Summing the geometrical series at the right-hand side results in the partition function for the quantum ME-packets of the form:

$$Z = \frac{\exp \left(\frac{\lambda_1^2}{4\lambda_3} + \frac{\lambda_2^2}{4\lambda_4} \right)}{2 \sinh(\hbar\sqrt{\lambda_3\lambda_4})}. \quad (3.24)$$

Now, we can express the Lagrange multipliers in terms of the expectation values and variances. Eqs. (3.12) and (3.13) yield

$$\lambda_1 = -\frac{Q}{\Delta Q^2} \frac{\nu}{2} \ln \frac{\nu+1}{\nu-1}, \quad \lambda_2 = -\frac{P}{\Delta P^2} \frac{\nu}{2} \ln \frac{\nu+1}{\nu-1}, \quad (3.25)$$

and

$$\lambda_3 = \frac{1}{2\Delta Q^2} \frac{\nu}{2} \ln \frac{\nu+1}{\nu-1}, \quad \lambda_4 = \frac{1}{2\Delta P^2} \frac{\nu}{2} \ln \frac{\nu+1}{\nu-1}, \quad (3.26)$$

where ν is defined by Eq. (2.1).

From Eqs. (3.14), (3.25) and (3.26), we obtain the entropy:

$$\Sigma = -\ln 2 + \frac{\nu+1}{2} \ln(\nu+1) - \frac{\nu-1}{2} \ln(\nu-1) . \quad (3.27)$$

Thus, Σ depends on Q , P , ΔQ , ΔP only via ν . We have

$$\frac{d\Sigma}{d\nu} = \frac{1}{2} \ln \frac{\nu+1}{\nu-1} > 0 ,$$

so that Σ is an increasing function of ν . Near $\nu = 1$,

$$\Sigma \approx -\frac{\nu-1}{2} \ln(\nu-1) .$$

Asymptotically ($\nu \rightarrow \infty$),

$$\Sigma \approx \ln \nu + 1 - \ln 2 .$$

It is clear that the choice of Q and P cannot influence the entropy. The independence of Σ from Q and P does not contradict the Legendre transformation properties. Indeed, usually, one would have

$$\frac{\partial \Sigma}{\partial Q} = \lambda_1 ,$$

but here

$$\frac{\partial \Sigma}{\partial Q} = \lambda_1 + 2\lambda_3 Q ,$$

which is zero.

Eq. (3.16) implies that

$$-2\sqrt{\lambda_3\lambda_4}\mathbf{K} = -\lambda_3 \left(\mathbf{q} + \frac{\lambda_1}{2\lambda_3} \right)^2 - \lambda_4 \left(\mathbf{p} + \frac{\lambda_2}{2\lambda_4} \right)^2 .$$

Substituting for the Lagrange multipliers from Eqs. (3.25) and (3.26), we obtain

$$-2\sqrt{\lambda_3\lambda_4}\mathbf{K} = -\frac{\nu}{2} \ln \frac{\nu+1}{\nu-1} \left[\frac{1}{2} \frac{(\mathbf{q}-Q)^2}{\Delta Q^2} + \frac{1}{2} \frac{(\mathbf{p}-P)^2}{\Delta P^2} \right] .$$

The resulting state operator, generalised to n degrees of freedom, is then described by the following

Theorem 3.1 *The state operator $\mathbb{T}[Q, P, \Delta Q, \Delta P]$ of the ME packet of a system with n degrees of freedom for given expectation values and variances Q_1, \dots, Q_n , $\Delta Q_1, \dots, \Delta Q_n$ of coordinates and P_1, \dots, P_n , $\Delta P_1, \dots, \Delta P_n$ of momenta, is*

$$\mathbb{T}[Q, P, \Delta Q, \Delta P] = \prod_{k=1}^n \left[\frac{2}{\sqrt{\nu_k^2 - 1}} \exp \left(-\frac{\nu_k}{2} \ln \frac{\nu_k + 1}{\nu_k - 1} \mathbf{K}'_k \right) \right] , \quad (3.28)$$

where

$$K'_k = \frac{1}{2} \frac{(\mathbf{q}_k - Q_k)^2}{\Delta Q_k^2} + \frac{1}{2} \frac{(\mathbf{p}_k - P_k)^2}{\Delta P_k^2} \quad (3.29)$$

and

$$\nu_k = \frac{2\Delta P_k \Delta Q_k}{\hbar} . \quad (3.30)$$

It may be interesting to observe that, strictly speaking, the state operator (3.28) is not a Gaussian distribution. Thus, it seems to be either a counterexample to, or a generalization of, Jaynes' hypothesis (see the remark after Theorem 2.1).

Let us study some further properties of quantum ME packets. In the diagonal representation, we have for one degree of freedom, $n = 1$:

$$\mathsf{T}[Q, P, \Delta Q, \Delta P] = \sum_{m=0}^{\infty} R_m |m\rangle \langle m| . \quad (3.31)$$

We easily obtain

$$R_m = \langle m | \mathsf{T}[Q, P, \Delta Q, \Delta P] | m \rangle = 2 \frac{(\nu - 1)^m}{(\nu + 1)^{m+1}} . \quad (3.32)$$

Hence,

$$\lim_{\nu \rightarrow 1} R_m = \delta_{m0} ,$$

and the state $\mathsf{T}[Q, P, \Delta Q, \Delta P]$ becomes $|0\rangle \langle 0|$. In general, states $|m\rangle$ depend on ν . The state vector $|0\rangle$ expressed as a function of $Q, P, \Delta Q$ and ν is given, for any ν , by

$$\psi(q) = \left(\frac{1}{\pi} \frac{\nu}{2\Delta Q^2} \right)^{1/4} \exp \left[-\frac{\nu}{4\Delta Q^2} (q - Q)^2 + \frac{iPq}{\hbar} \right] . \quad (3.33)$$

This is a Gaussian wave packet that corresponds to different values of variances than ME packet (3.31): these values satisfy the minimum uncertainty condition. For $\nu \rightarrow 1$, it remains regular and the projection $|0\rangle \langle 0|$ becomes the state operator of the original ME packet. Hence, Gaussian wave packets are special cases of quantum ME packets.

The diagonal representation offers a method for calculating expectation values of coordinates and momenta products in a quantum ME-packet state that can replace formula (2.11). Let us denote such a product X . We have

$$\langle X \rangle = \sum_{k=0}^{\infty} R_k \langle k | X | k \rangle . \quad (3.34)$$

To calculate $\langle k | X | k \rangle$, we use Eqs. (3.19), (3.20) and (3.17) to obtain

$$\mathbf{q} = Q + \frac{\Delta Q}{\sqrt{\nu}} (\mathbf{A} + \mathbf{A}^\dagger) , \quad \mathbf{p} = P - i \frac{\Delta P}{\sqrt{\nu}} (\mathbf{A} - \mathbf{A}^\dagger) . \quad (3.35)$$

After substituting these relations into \mathbf{X} , \mathbf{X} can be expressed as a polynomial in \mathbf{A} and \mathbf{A}^\dagger that will be denoted by $X(\mathbf{A}, \mathbf{A}^\dagger)$. Now, we define a map \mathcal{N} such that $\mathcal{N}(X(\mathbf{A}, \mathbf{A}^\dagger))$ is a polynomial of the single variable $\mathbf{A}^\dagger \mathbf{A}$ in two steps.

1. All monomials in $X(\mathbf{A}, \mathbf{A}^\dagger)$ that contain different numbers of \mathbf{A} and \mathbf{A}^\dagger factors are discarded. A polynomial $\bar{X}(\mathbf{A}, \mathbf{A}^\dagger)$ results.
2. Using the commutation relations (3.18), each monomial in $\bar{X}(\mathbf{A}, \mathbf{A}^\dagger)$ is re-ordered so that it becomes a polynomial in a single variable $\mathbf{A}^\dagger \mathbf{A}$ and this is the desired $\mathcal{N}(X(\mathbf{A}, \mathbf{A}^\dagger))$, which will be denoted by $X_{\mathcal{N}}(\mathbf{A}^\dagger \mathbf{A})$.

It follows that \mathcal{N} is linear,

$$\mathcal{N}\left(X_1(\mathbf{A}, \mathbf{A}^\dagger) + X_2(\mathbf{A}, \mathbf{A}^\dagger)\right) = \mathcal{N}\left(X_1(\mathbf{A}, \mathbf{A}^\dagger)\right) + \mathcal{N}\left(X_2(\mathbf{A}, \mathbf{A}^\dagger)\right)$$

and commutes with \dagger ,

$$\mathcal{N}\left(X^\dagger(\mathbf{A}, \mathbf{A}^\dagger)\right) = \left(\mathcal{N}(X(\mathbf{A}, \mathbf{A}^\dagger))\right)^\dagger .$$

Examples:

$$\begin{aligned} \mathcal{N}(\mathbf{q}) &= Q , \\ \mathcal{N}(\mathbf{p}^2) &= P^2 + \frac{\Delta P^2}{\nu}(2\mathbf{A}^\dagger \mathbf{A} + 1) . \end{aligned}$$

Returning to the original task of calculating expectation value of X , we obtain

$$\langle k | \mathbf{X} | k \rangle = X_{\mathcal{N}}(k) .$$

In Eq. (3.34), there are, therefore, sums

$$\sum_{k=0}^{\infty} k^n R_k .$$

Substituting for R_k from Eq. (3.32), we arrive at

$$\sum_{k=0}^{\infty} k^n R_k = \frac{2}{\nu + 1} I_n ,$$

where

$$I_n(\nu) = \sum_{k=0}^{\infty} k^n \left(\frac{\nu - 1}{\nu + 1} \right)^k .$$

We easily obtain

$$I_n = \left(\frac{\nu^2 - 1}{2} \frac{d}{d\nu} \right)^n \frac{\nu + 1}{2} .$$

The desired expectation value value is then given by

$$\langle \mathbf{X} \rangle = \frac{1}{\nu + 1} X_{\mathcal{N}} \left(\frac{\nu^2 - 1}{2} \frac{d}{d\nu} \right) (\nu + 1) . \quad (3.36)$$

For example, we obtain:

$$\langle \mathbf{A}^\dagger \mathbf{A} \rangle = \frac{\nu}{2} - \frac{1}{2} , \quad (3.37)$$

$$\langle (\mathbf{A}^\dagger \mathbf{A})^2 \rangle = \frac{\nu^2}{2} - \frac{\nu}{2} , \quad (3.38)$$

$$\langle (\mathbf{A}^\dagger \mathbf{A})^3 \rangle = \frac{3\nu^3}{4} - \frac{3\nu^2}{4} - \frac{\nu}{4} + \frac{1}{4} . \quad (3.39)$$

The calculation of the polynomial $X_{\mathcal{N}}$ for a given \mathbf{X} and the evaluation of the right-hand side of Eq. (3.36) are the two steps of the promised method.

3.3 Polynomial potential function

Let the Hamiltonian of S_q be that of Eq. (2.4) and the unitary evolution group be $\mathbf{U}(t)$. The dynamics in the Schrödinger picture leads to the time dependence of \mathbf{T} :

$$\mathbf{T}(t) = \mathbf{U}(t) \mathbf{T} \mathbf{U}(t)^\dagger .$$

Substituting for \mathbf{T} from Eq. (3.28) and using a well-known property of exponential functions, we obtain

$$\mathbf{T}(t) = \frac{2}{\sqrt{\nu^2 - 1}} \exp \left(-\frac{\nu}{2} \ln \frac{\nu + 1}{\nu - 1} \mathbf{U}(t) \mathbf{K}' \mathbf{U}(t)^\dagger \right) . \quad (3.40)$$

As \mathbf{K}' is not a Hamiltonian of S_q , $\mathbf{U}(t) \mathbf{K}' \mathbf{U}(t)^\dagger$ is difficult to calculate.

In the Heisenberg picture, \mathbf{T} remains constant, while observables are time dependent. We denote such time-dependent position and momentum operators by $\bar{\mathbf{q}}$ and $\bar{\mathbf{p}}$ to distinguish them from their initial values \mathbf{q} and \mathbf{p} at $t = 0$. Operators $\bar{\mathbf{q}}$ and $\bar{\mathbf{p}}$ satisfy the equations

$$i\hbar \frac{d\bar{\mathbf{q}}}{dt} = [\bar{\mathbf{q}}, \mathbf{H}] , \quad i\hbar \frac{d\bar{\mathbf{p}}}{dt} = [\bar{\mathbf{p}}, \mathbf{H}] , \quad (3.41)$$

which are solved by

$$\bar{\mathbf{q}}(t; \mathbf{q}, \mathbf{p}) = \mathbf{U}(t) \mathbf{q} \mathbf{U}(t)^\dagger , \quad \bar{\mathbf{p}}(t; \mathbf{q}, \mathbf{p}) = \mathbf{U}(t) \mathbf{p} \mathbf{U}(t)^\dagger$$

for $\mathbf{q} = \bar{\mathbf{q}}(0; \mathbf{q}, \mathbf{p})$ and $\mathbf{p} = \bar{\mathbf{p}}(0; \mathbf{q}, \mathbf{p})$. The resulting operators can be written in the form of operator functions analogous to classical expressions (2.15) so that Eqs. (2.17) and (2.18) can again be used in the operator form if q and p are replaced by \mathbf{q} and \mathbf{p} .

Let us assume that the potential function is given by Eq. (2.31) in the operator form:

$$V(\bar{\mathbf{q}}) = \sum_{k=1}^N \frac{1}{k!} V_k \bar{\mathbf{q}}^k .$$

Heisenberg equations of motion (2.5) that result from Eq. (3.41),

$$\frac{\partial \bar{\mathbf{q}}}{\partial t} = \frac{1}{\mu} \bar{\mathbf{p}} , \quad (3.42)$$

$$\frac{\partial \bar{\mathbf{p}}}{\partial t} = -V'(\bar{\mathbf{q}}) , \quad (3.43)$$

can be used to calculate all time derivatives of the functions $\bar{\mathbf{q}}$ and $\bar{\mathbf{p}}$. Here, the symbol $X'(\mathbf{x})$ denotes the derivative of the polynomial $X(\mathbf{x})$, for example,

$$V'(\mathbf{q}) = \sum_{k=0}^N \frac{1}{k!} V_{k+1} \mathbf{q}^k .$$

For the expectation values and variances in a ME-packet state, we shall use the same notation as in the classical case, that is,

$$\bar{Q}(t) = \langle \bar{\mathbf{q}} \rangle , \quad \bar{P}(t) = \langle \bar{\mathbf{p}} \rangle , \quad \Delta \bar{Q}(t) = \Delta \bar{\mathbf{q}} , \quad \Delta \bar{P}(t) = \Delta \bar{\mathbf{p}} ,$$

so that $Q = \bar{Q}(0)$, etc. In the Heisenberg picture, expectation values of time derivatives are time derivatives of expectation values. For instance, we have

$$\left(\frac{d\bar{Q}}{dt} \right)_0 = \left\langle \frac{d\bar{\mathbf{q}}}{dt} \right\rangle_0 ,$$

the index 0 indicating the value taken at $t = 0$. Then, we can calculate all time derivatives of expectation values \bar{Q} , \bar{P} and variances $\Delta \bar{Q}$ and $\Delta \bar{P}$ at $t = 0$ similarly as in Section 2.4. In the case of variances, we obtain from the definition of variance ($\Delta O = \sqrt{\langle O^2 \rangle - \langle O \rangle^2}$, see [1], p. 223):

$$\frac{\partial}{\partial t} \Delta \bar{\mathbf{q}} = \frac{1}{2\Delta \bar{\mathbf{q}}} \frac{\partial}{\partial t} (\langle \bar{\mathbf{q}}^2 \rangle - \langle \bar{\mathbf{q}} \rangle^2)$$

and

$$\frac{\partial}{\partial t} \Delta \bar{\mathbf{p}} = \frac{1}{2\Delta \bar{\mathbf{p}}} \frac{\partial}{\partial t} (\langle \bar{\mathbf{p}}^2 \rangle - \langle \bar{\mathbf{p}} \rangle^2) .$$

Eqs. (3.42) and (3.43) imply that

$$\frac{\partial \Delta \bar{Q}}{\partial t} = \frac{1}{2\mu \Delta \bar{Q}} (\langle \bar{\mathbf{q}} \bar{\mathbf{p}} + \bar{\mathbf{p}} \bar{\mathbf{q}} \rangle - 2\langle \bar{\mathbf{q}} \rangle \langle \bar{\mathbf{p}} \rangle) \quad (3.44)$$

and

$$\frac{\partial \Delta \bar{P}}{\partial t} = -\frac{1}{\Delta \bar{P}} \langle \bar{\mathbf{p}} V'(\bar{\mathbf{q}}) + V'(\bar{\mathbf{q}}) \bar{\mathbf{p}} \rangle + 2\langle \bar{\mathbf{p}} \rangle \langle V'(\bar{\mathbf{q}}) \rangle . \quad (3.45)$$

To calculate any further, we need some properties of expectation values of products of \mathbf{q} and \mathbf{p} .

Lemma 3.3 *Let X_{mn} be a product of m factors \mathbf{q} and n factors \mathbf{p} in some ordering. Let $\langle X_{mn} + X_{mn}^\dagger \rangle$ be the expectation value in the ME packet defined by Q , P , ΔQ and ΔP . Then*

$$\langle X_{mn} + X_{mn}^\dagger \rangle = \bar{X}_{mn} + \bar{X}_{mn}^\dagger , \quad (3.46)$$

where

$$\bar{X}_{mn} = \sum_{u=0}^{[m/2]} \sum_{v=0}^{[n/2]} (-1)^v Q^{m-2u} P^{n-2v} \Delta Q^{2u} \Delta P^{2v} \nu^{-u-v} \langle Y_{(2u)(2v)}(\mathbf{A} + \mathbf{A}^\dagger, \mathbf{A} - \mathbf{A}^\dagger) \rangle \quad (3.47)$$

and $Y_{(2u)(2v)}(\mathbf{A} + \mathbf{A}^\dagger, \mathbf{A} - \mathbf{A}^\dagger)$ is a product of $2u$ factors $\mathbf{A} + \mathbf{A}^\dagger$ and $2v$ factors $\mathbf{A} - \mathbf{A}^\dagger$ in some ordering multiplied by some integer coefficient.

Proof According to the method of diagonal representation of Section 3.2, we have to use Eq. (3.35) so that $\langle X_{mn} \rangle$ becomes

$$\langle X_{mn} \rangle = \sum_{x=0}^m \sum_{y=0}^n (-i)^y Q^{m-x} P^{n-y} \Delta Q^x \Delta P^y \nu^{-(x+y)/2} \langle Y_{xy}(\mathbf{A} + \mathbf{A}^\dagger, \mathbf{A} - \mathbf{A}^\dagger) \rangle , \quad (3.48)$$

where $Y_{xy}(\mathbf{A} + \mathbf{A}^\dagger, \mathbf{A} - \mathbf{A}^\dagger)$ is a product of x factors $\mathbf{A} + \mathbf{A}^\dagger$ and y factors $\mathbf{A} - \mathbf{A}^\dagger$ in some ordering.

The second step is to transform Y_{xy} to a polynomial in \mathbf{A} and \mathbf{A}^\dagger and discard all terms in which the number of factors \mathbf{A} differs from that of \mathbf{A}^\dagger . The result is symbolised by $Y_{xy}(\mathbf{A} + \mathbf{A}^\dagger, \mathbf{A} - \mathbf{A}^\dagger) \mapsto Y_{xy}^{(0)}(\mathbf{A}, \mathbf{A}^\dagger)$. Clearly, $Y_{xy}^{(0)}(\mathbf{A}, \mathbf{A}^\dagger) = 0$ if $x + y$ is odd. Hence, only terms with even $x + y$ contribute to the expectation value.

The third step is to reorder $Y_{xy}^{(0)}(\mathbf{A}, \mathbf{A}^\dagger)$ using the commutation relation (3.18) so that we obtain

$$Y_{xy}^{(0)}(\mathbf{A}, \mathbf{A}^\dagger) = Y_{xy}^{(1)}(\mathbf{A}^\dagger \mathbf{A}) ,$$

where

$$Y_{xy}^{(1)}(\mathbf{A}^\dagger \mathbf{A}) = \sum_{k=0}^{(x+y)/2} z_k (\mathbf{A}^\dagger \mathbf{A})^k .$$

It is easy to see that z_k are integers. Operator $Y_{xy}^{(1)}(\mathbf{A}^\dagger \mathbf{A})$ is therefore self-adjoint. It follows that the terms in Eq. (3.48) with odd y 's cancel in $X_{mn} + X_{mn}^\dagger$.

QED.

As an example, we use Lemma 3.3 to simplify some terms in Eqs. (3.44) and (3.45):

$$\langle \bar{\mathbf{q}} \bar{\mathbf{p}} + \bar{\mathbf{p}} \bar{\mathbf{q}} \rangle_0 = 2QP \quad (3.49)$$

and

$$\langle \bar{\mathbf{p}} V'(\bar{\mathbf{q}}) + V'(\bar{\mathbf{q}})(\bar{\mathbf{p}}) \rangle_0 = 2P \langle V'(\bar{\mathbf{q}}) \rangle_0$$

so that finally

$$\left(\frac{d\Delta\bar{Q}}{dt}\right)_0 = \left(\frac{d\Delta\bar{P}}{dt}\right)_0 = 0 . \quad (3.50)$$

We then also have, as in the classical case:

$$\left(\frac{d^K\Delta\bar{Q}}{dt^K}\right)_0 = \frac{1}{2\Delta Q} \left(\frac{d^K(\langle\bar{q}^2\rangle - \langle\bar{q}\rangle^2)}{dt^K}\right)_0 ,$$

and

$$\left(\frac{d^K\Delta\bar{P}}{dt^K}\right)_0 = \frac{1}{2\Delta P} \left(\frac{d^K(\langle\bar{p}^2\rangle - \langle\bar{p}\rangle^2)}{dt^K}\right)_0 .$$

The case of potential function (2.19) is solvable in quantum theory exactly as in the classical one, and we can use it for comparison with the classical dynamics as well as for a better understanding of the ME-packet dynamics. Eqs. (3.41) have then the solutions given by (2.20) and (2.21) with functions $f_n(t)$ and $g_n(t)$ given by (2.22) and (2.23) or (2.24) and (2.25). The calculation of the expectation values and variances is analogous to the classical one and we obtain Eqs. (2.26) and Eq. (2.28) again with the difference that the term $2\langle qp \rangle$ on the right hand side of (2.27) is now replaced by $\langle \mathbf{q}\mathbf{p} + \mathbf{p}\mathbf{q} \rangle$, which is given by Eq. (3.49). The result is again Eq. (2.28). Similarly for \mathbf{p} , the results are given by Eqs. (2.29) and (2.30). Hence, quantum and classical dynamics coincide for the polynomial potentials of the second and lower orders.

For a general polynomial potential V , we can calculate all time derivatives of \bar{Q} , \bar{P} , $\Delta\bar{Q}$, $\Delta\bar{P}$ by an iterative application of Eqs. (3.42) and (3.43). For example,

$$\left(\frac{d\bar{P}}{dt}\right)_0 = -\langle V' \rangle , \quad (3.51)$$

$$\left(\frac{d^2\bar{P}}{dt^2}\right)_0 = -\langle (V')^t \rangle , \quad (3.52)$$

where $(V')^t$ is obtained from the polynomial V' of \mathbf{q} by discarding constant terms and replacing each \mathbf{q} by \mathbf{p}/μ one by one. For example, if $N = 4$,

$$(V')^t = \frac{1}{\mu} \left[V_2 \mathbf{p} + \frac{V_3}{2} (\mathbf{q}\mathbf{p} + \mathbf{p}\mathbf{q}) + \frac{V_4}{6} (\mathbf{q}^2 \mathbf{p} + \mathbf{q}\mathbf{p}\mathbf{q} + \mathbf{p}\mathbf{q}^2) \right] .$$

The time derivatives of $Q(t)$ are then determined by the formula:

$$\left(\frac{d^k\bar{Q}}{dt^k}\right)_0 = \left\langle \frac{1}{\mu} \frac{d^{k-1}\bar{P}}{dt^{k-1}} \right\rangle_0 .$$

In an analogous way, we obtain for the second time derivatives of the variances:

$$\left(\frac{d^2\Delta\bar{Q}}{dt^2}\right)_0 = \frac{1}{\mu\Delta Q} (-\langle \mathbf{q}V' \rangle - \langle \mathbf{q} \rangle \langle V' \rangle + \Delta P^2) \quad (3.53)$$

and

$$\left(\frac{d^2\Delta\bar{P}}{dt^2}\right)_0 = \frac{1}{2\Delta P}[2\langle V'^2\rangle - 2\langle V'\rangle^2 - \langle(V')^t\mathbf{p} + \mathbf{p}(V')^t\rangle + 2\langle(V')^t\rangle\langle\mathbf{p}\rangle] . \quad (3.54)$$

An important property of Eq. (3.53) is the proportionality of the right-hand side to μ^{-1} . Clearly, this property will be preserved for all higher time derivatives because they are calculated by time derivatives of Eq. (3.53). Such derivatives are then applied to various polynomials of \mathbf{q} and \mathbf{p} and lead at most to further negative powers of μ . The proportionality to μ^{-1} is confirmed by Eqs. (2.28) and by motion of Gaussian wave packets. It also holds for classical evolution of position variance. Hence, the spreading of macroscopic ME packets (including Gaussian wave packets) can be very slow in general.

The next step is to calculate the expectation values of the products of \mathbf{q} and \mathbf{p} that occur in time derivatives of the averages and variances. As we are mainly interested in differences between the quantum and classical equations, we list only the lowest-order products that show non-trivial quantum corrections:

$$\langle\mathbf{q}^6\rangle = Q^6 + 15Q^4\Delta Q^2 + 45Q^2\Delta Q^4 + 15\Delta Q^6 + 9\Delta Q^6\nu^{-1} - 3\Delta Q^6\nu^{-3} , \quad (3.55)$$

$$\begin{aligned} \langle\mathbf{q}^2\mathbf{p}^2 + \mathbf{p}^2\mathbf{q}^2\rangle &= 2Q^2P^2 + 2Q^2\Delta P^2 + 2P^2\Delta Q^2 + 2\Delta Q^2\Delta P^2 \\ &\quad - 4\Delta Q^2\Delta P^2\nu^{-2} , \end{aligned} \quad (3.56)$$

$$\langle\mathbf{p}\mathbf{q}^2\mathbf{p}\rangle = Q^2P^2 + Q^2\Delta P^2 + P^2\Delta Q^2 + \Delta Q^2\Delta P^2 + 2\Delta Q^2\Delta P^2\nu^{-2} , \quad (3.57)$$

$$\langle\mathbf{q}\mathbf{p}^2\mathbf{q}\rangle = Q^2P^2 + Q^2\Delta P^2 + P^2\Delta Q^2 + \Delta Q^2\Delta P^2 + 2\Delta Q^2\Delta P^2\nu^{-2} . \quad (3.58)$$

From these formulas, we can also infer quantum corrections of one degree higher products because of Lemma 3.3. For example, $\langle\mathbf{q}^7\rangle = Q\langle\mathbf{q}^6\rangle$, etc.

In [25], the first four time derivatives of \bar{Q} and \bar{P} at $t = 0$ have been calculated for fourth-order potential functions and shown to coincide with the corresponding classical expressions. A more interesting question is, at which degree of the polynomial potential any quantum corrections appear in a time derivative of some order higher than 4. One way to answer the question is to look for the lowest time derivative of \bar{P} that contains the \mathbf{q}^6 term. A short estimate shows that it is the 9-th derivative and a lengthy calculation yields the explicit form of the term:

$$\left(\frac{d^9\bar{P}}{dt^9}\right)_0 = \dots - \frac{125}{4} \frac{V_3^5}{\mu^4} \mathbf{q}^6 .$$

Hence, quantum dynamics differs from the classical one for polynomial potentials of degree 3 and higher. This is the same number as that obtained in Section 2.3, but the reason is now rather different from that considered there, the corrections are different and appear at higher time derivatives.

3.4 The Theorem

Eqs. (3.55) - (3.58) show that the quantum corrections to the expectation values of **qp**-products, at least for the products considered there, have the following structure: the corrections are proportional to ν^{-1} and are multiplied by factors of the form $\Delta Q^a \Delta P^b$, where a and b are some integers. The classical expression that is corrected always contains at least one term multiplied by the same factor $\Delta Q^a \Delta P^b$. Such a structure suggests that the quantum corrections become negligible with respect to the corrected classical expression in the limit $\Delta Q \rightarrow \infty$, $\Delta P \rightarrow \infty$, at least in the cases listed, because ν itself is proportional to $\Delta Q \Delta P$.

The aim of the present section is to prove that this structure is a general property of polynomial potential functions of arbitrary high degree and of arbitrary high time derivatives of the expectation values and variances.

Theorem 3.2 *Let X_{mn} be a product of n factors of **q** and m factors of **p** in an arbitrary ordering. Let $\langle X_{mn} + X_{mn}^\dagger \rangle_q$ be the expectation value in the quantum ME packet and $\langle q^m p^n \rangle_c$ that in the classical ME packet, both packets being defined by the same values of Q , P , ΔQ and ΔP . Then, first,*

$$\langle X_{mn} + X_{mn}^\dagger \rangle_q = 2\langle q^m p^n \rangle_c + 2R[X_{mn}](Q, P, \Delta Q, \Delta P) , \quad (3.59)$$

where $R[X_{mn}](Q, P, \Delta Q, \Delta P)$ is a function of Q , P , ΔQ and ΔP depending on \hbar via ν . Second,

$$\lim_{\Delta Q \rightarrow \infty, \Delta P \rightarrow \infty} \frac{R[X_{mn}](Q, P, \Delta Q, \Delta P)}{\langle q^m p^n \rangle_c} = 0 . \quad (3.60)$$

Proof To show Eq. (3.59), we first compare the quantum partition function (3.24) with its classical counterpart (2.10). They differ by the denominators $\sinh(\hbar\sqrt{\lambda_3\lambda_4})$ and $\hbar\sqrt{\lambda_3\lambda_4}$. If

$$\hbar\sqrt{\lambda_3\lambda_4} \ll 1 , \quad (3.61)$$

we can write

$$\sinh(\hbar\sqrt{\lambda_3\lambda_4}) = \hbar\sqrt{\lambda_3\lambda_4}[1 + O((\hbar\sqrt{\lambda_3\lambda_4})^2)]$$

The leading term in the quantum partition function then is

$$Z = \frac{\pi}{h} \frac{1}{\sqrt{\lambda_3\lambda_4}} \exp\left(\frac{\lambda_1^2}{4\lambda_3} + \frac{\lambda_2^2}{4\lambda_4}\right) .$$

Comparing this with formula (2.10) shows that the two expressions coincide if we set

$$v = h ,$$

where $\hbar = 2\pi\hbar$. Thus, quantum mechanics suggests a value for ν . Next, we have to express condition (3.61) in terms of the expectation values and variances. Eqs. (3.25) and (3.26) imply

$$\hbar\sqrt{\lambda_3\lambda_4} = \frac{1}{2} \ln \frac{\nu+1}{\nu-1} .$$

Hence, condition (3.61) is equivalent to

$$\nu \gg 1 . \quad (3.62)$$

It follows from the above consideration that the terms in the quantum expectation values that are of the lowest order in \hbar come from that part of the quantum partition function that has the same form as the classical partition function. It is true that the quantum expectation values cannot be obtained from the quantum partition function by the same way as classical expectation values are from the classical partition function. However, as it follows from the proof of Lemma 3.2, the terms of the lowest order in \hbar can be obtained so. Hence, the leading terms in the quantum expectation values have the same form as the classical terms. Then, if we replace \hbar by $2\Delta Q\Delta P/\nu$, Eq. (3.59) will result.

To prove Eq. (3.60), let us write the contribution of uv -term to $\bar{X}_{mn} + \bar{X}_{mn}^\dagger$ given by Eq. (3.47) as follows

$$(\bar{X}_{mn} + \bar{X}_{mn}^\dagger)_{uv} = 2(-1)^v Q^{m-2u} P^{n-2v} \Delta Q^{2u} \Delta P^{2v} \nu^{-u-v} \langle Y_{(2u)(2v)}^{(1)} (\mathbf{A}^\dagger \mathbf{A}) \rangle .$$

To proceed, we need the following Lemma.

Lemma 3.4 *For any positive integer k ,*

$$\langle (\mathbf{A}^\dagger \mathbf{A})^k \rangle = \frac{k!}{2^k} \nu^k + h_k(\nu) ,$$

where h_k is a polynomial of degree $k-1$.

Proof We use the method of mathematical induction. First,

$$\langle \mathbf{A}^\dagger \mathbf{A} \rangle = \frac{1}{2} \nu - \frac{1}{2} .$$

Second, according to Eq. (3.36),

$$\begin{aligned} \langle (\mathbf{A}^\dagger \mathbf{A})^{k+1} \rangle &= \frac{1}{\nu+1} \left(\frac{\nu^2-1}{2} \frac{d}{d\nu} \right)^{k+1} (\nu+1) \\ &= \frac{\nu-1}{2} \frac{d}{d\nu} \left[(1+\nu) \frac{1}{1+\nu} \left(\frac{\nu^2-1}{2} \frac{d}{d\nu} \right)^k (\nu+1) \right] = \frac{\nu-1}{2} \frac{d}{d\nu} [(1+\nu) \langle (\mathbf{A}^\dagger \mathbf{A})^k \rangle] \\ &= \frac{\nu-1}{2} \frac{d}{d\nu} \left[(1+\nu) \left(\frac{k!}{2^k} \nu^k + h_k(\nu) \right) \right] = \frac{(k+1)!}{2^{k+1}} \nu^{k+1} + h_{k+1} , \end{aligned}$$

QED.

Using the Lemma, we obtain

$$(\bar{X}_{mn} + \bar{X}_{mn}^\dagger)_{uv} = 2Q^{m-2u}P^{n-2v}\Delta Q^{2u}\Delta P^{2v} \left[(-1)^v z_{u+v} \frac{(u+v)!}{2^{u+v}} + \frac{1}{\nu^{u+v}} h_{u+v}^{(1)}(\nu) \right] ,$$

where

$$h_{u+v}^{(1)}(\nu) = \sum_{k=0}^{u+v-1} z_k \left(\frac{k!}{2^k} + h_k(\nu) \right) + z_{u+v} h_{u+v}(\nu)$$

is a polynomial of ν of degree $u + v - 1$. The first term in the brackets does not contain \hbar while the second is proportional to \hbar . Hence, the first term is the classical part and the second is the quantum correction. If $z_{u+v} \neq 0$, the classical part is multiplied by the same factor $\Delta Q^{2u}\Delta P^{2v}$ as the quantum correction to it is.

Suppose that $z_{u+v} = 0$. Then, the quantum corrections would be multiplied by a higher power of ΔQ and ΔP than the corrected classical part. But Lemma 2.1 implies that the corresponding term cannot disappear in the classical part. This holds for any u and v .

The term of the highest power of $\Delta Q\Delta P$ in Eq. (3.47) that gives a non-zero contribution to the quantum expectation value is $\Delta Q^{2u}\Delta P^{2v}$ with

$$u = \left\lfloor \frac{m}{2} \right\rfloor , \quad v = \left\lfloor \frac{n}{2} \right\rfloor .$$

The above argument shows that the highest power of $\Delta Q\Delta P$ in the quantum correction to the expectation value cannot be higher. Then, Eq. (3.60) follows immediately, **QED.**

Theorem 3.2 motivates the following:

High-Entropy Conjecture *For all reasonable potentials, the classical and quantum trajectories of ME packets satisfy:*

$$\lim_{\Delta Q \rightarrow \infty, \Delta P \rightarrow \infty} \frac{Q_q(t) - Q_c(t)}{Q_c(t)} = 0 , \quad \lim_{\Delta Q \rightarrow \infty, \Delta P \rightarrow \infty} \frac{P_q(t) - P_c(t)}{P_c(t)} = 0 ,$$

and

$$\lim_{\Delta Q \rightarrow \infty, \Delta P \rightarrow \infty} \frac{\Delta Q_q(t) - \Delta Q_c(t)}{\Delta Q_c(t)} = 0 , \quad \lim_{\Delta Q \rightarrow \infty, \Delta P \rightarrow \infty} \frac{\Delta P_q(t) - \Delta P_c(t)}{\Delta P_c(t)} = 0 ,$$

for all t for which the formulas make sense.

That is: the fuzzier the classical and quantum ME packets are, the closer their trajectories are to each other. But the entropy of an ME packet is an increasing function of $\Delta Q\Delta P$. One can therefore say that the classical limit is a high-entropy

limit for mechanics, which is similar to what is, in certain sense, also true for thermodynamics.

High-Entropy Conjecture has been proven only for polynomial potential functions. More models ought to be studied, for example the Coulomb potential and the corresponding theory of Kepler orbits.

3.5 A model of classical rigid body

To show how the above theory of classical properties works, we construct a one dimensional model of a free rigid body. Large parts of this section follow [35]. The restrictions to one dimension and absence of external forces enable us to calculate everything explicitly—the model is completely solvable. The real object \mathcal{S} is a very thin and very stiff solid rod free to move in the direction of its length. Its classical model S_c is a one-dimensional continuum of mass M and length L . Classical observables are internal energy E , temperature, centre of mass position X , total momentum P and their variances ΔX and ΔP . The one-dimensionality, zero forces, the length and the total mass (structural parameters) are just part of the definition of the model valid for any state of it.

The construction of its quantum model S_q entails that, first, the structural properties of the system must be given, second, its Hilbert space and observables defined, third, the objective properties specified that correspond to all classical properties and fourth, the suitable states are to be determined by the preparation defined by the values E , X , P , ΔX and ΔP .

Assumption 3.1 S_q is a linear chain of $N+1$ particles of mass μ distributed along the x -axis with the Hamiltonian

$$H = \frac{1}{2\mu} \sum_{n=1}^{N+1} \mathbf{p}^{(n)2} + \frac{\kappa^2}{2} \sum_{n=2}^{N+1} (\mathbf{x}^{(n)} - \mathbf{x}^{(n-1)} - \xi)^2, \quad (3.63)$$

involving only nearest-neighbour elastic forces. Here operator $\mathbf{x}^{(n)}$ is the position, operator $\mathbf{p}^{(n)}$ the momentum of the n -th particle, κ the oscillator strength and ξ the equilibrium interparticle distance.

The parameters N , μ , κ , ξ and the label n are structural properties. The particles are not identical but distinguished by their ordering within the chain. The minimal set of observables that generate the whole algebra of observables of S_q are $\mathbf{x}^{(n)}$ and $\mathbf{p}^{(n)}$, $n = 1, \dots, N$. Their number is $2N + 2$. Thus, the quantum model is much richer than the classical one, whose state is determined just by 5 numbers. We shall construct some set of quantum observables and obtain some relations between structural parameters of the two models, such as between μ and M or ξ and L .

This kind of chain is in one respect different from most chains that are studied in literature: the positions of the chain particles are dynamical variables so that the chain can move as a whole and the invariance with respect to (one-dimensional space) Galilean group is not disturbed. However, the chain can still be solved by methods that are described in [42, 65].

First, we find the variables \mathbf{u}_n and \mathbf{q}_n that diagonalize the Hamiltonian and define thus the so-called normal modes. The transformation is

$$\mathbf{x}^{(n)} = \sum_{m=0}^N Y_n^m \mathbf{u}_m + \left(n - \frac{N+2}{2}\right) \xi , \quad (3.64)$$

and

$$\mathbf{p}^{(n)} = \sum_{m=0}^N Y_n^m \mathbf{q}_m , \quad (3.65)$$

where the mode index m runs through $0, 1, \dots, N$ and Y_n^m is an orthogonal matrix; for even m ,

$$Y_n^m = A(m) \cos \left[\frac{\pi m}{N} \left(n - \frac{N+2}{2} \right) \right] , \quad (3.66)$$

while for odd m ,

$$Y_n^m = A(m) \sin \left[\frac{\pi m}{N} \left(n - \frac{N+2}{2} \right) \right] , \quad (3.67)$$

and the normalization factors are given by

$$A(0) = \frac{1}{\sqrt{N+1}} , \quad A(m) = \sqrt{\frac{2}{N+1}} , \quad m > 0 . \quad (3.68)$$

To show that \mathbf{u}_n and \mathbf{q}_n do represent normal modes, we substitute Eqs. (3.64) and (3.65) into (3.63) and obtain, after some calculation,

$$H = \frac{1}{2\mu} \sum_{m=0}^N \mathbf{q}_m^2 + \frac{\mu}{2} \sum_{m=0}^N \omega_m^2 \mathbf{u}_m^2 ,$$

which is indeed diagonal. The mode frequencies are

$$\omega_m = \frac{2\kappa}{\sqrt{\mu}} \sin \frac{m\pi}{N+2} . \quad (3.69)$$

Consider the terms with $m = 0$. We have $\omega_0 = 0$, and $Y_n^0 = 1/\sqrt{N+1}$. Hence,

$$\mathbf{u}_0 = \sum_{n=1}^{N+1} \frac{1}{\sqrt{N+1}} \mathbf{x}^{(n)} , \quad \mathbf{q}_0 = \sum_{n=1}^{N+1} \frac{1}{\sqrt{N+1}} \mathbf{p}^{(n)} ,$$

so that

$$u_0 = X\sqrt{N+1} , \quad q_0 = \frac{P}{\sqrt{N+1}} ,$$

where X is the centre-of-mass coordinate of the chain and P is its total momentum. The “zero” terms in the Hamiltonian then reduce to

$$\frac{1}{2(N+1)\mu} P^2 . \quad (3.70)$$

Thus, the “zero mode” describes a straight, uniform motion of the chain as a whole. The fact that the centre of mass degrees of freedom decouple from other, internal, ones is a consequence of the Galilean invariance.

The other modes are harmonic oscillators called “phonons” with frequencies ω_m , $m = 1, 2, \dots, N$. Important observables are the energy of the phonons,

$$H_{\text{int}} = \frac{1}{2\mu} \sum_{m=1}^N q_m^2 + \frac{\mu}{2} \sum_{m=1}^N \omega_m^2 u_m^2 , \quad (3.71)$$

and the length of the body,

$$L = x^{(N)} - x^{(1)} . \quad (3.72)$$

We assume that

$$E = \langle H_{\text{int}} \rangle , \quad M = (N+1)\mu ,$$

$$L = \langle L \rangle , \quad X = \langle X \rangle , \quad P = \langle P \rangle , \quad \Delta X = \langle \Delta X \rangle , \quad \Delta P = \langle \Delta P \rangle .$$

The next point is the choice of suitable states. We write the Hilbert space of S_q as

$$\mathbf{H} = \mathbf{H}_{\text{CM}} \otimes \mathbf{H}_{\text{int}} ,$$

where \mathbf{H}_{CM} is constructed from the wave functions $\Psi(X)$ and \mathbf{H}_{int} has the phonon eigenstates as a basis.

Assumption 3.2 *The suitable states have the form*

$$\mathbf{T}_{\text{CM}} \otimes \mathbf{T}_{\text{int}} .$$

Internal state \mathbf{T}_{int} maximises the entropy under the condition of fixed expectation value of the internal energy,

$$\text{Tr}(\mathbf{T}_{\text{int}} \mathbf{H}_{\text{int}}) = E , \quad (3.73)$$

where Tr is the partial trace over \mathbf{H}_{int} . The external state \mathbf{T}_{CM} is the ME packet for given expectation values X , P , ΔX and ΔP .

For T_{int} , the condition of maximum entropy has to do with the preparation. Physically, the thermodynamic equilibrium can settle down spontaneously starting from an arbitrary state only if some weak but non-zero interaction exists both between the phonons and between the rod and the environment. We assume that this can be arranged so that the interaction can be neglected in the calculations of the present section. It turns then out that all other classical internal properties are functions of the classical internal energy.

The mathematics associated with the maximum entropy principle is variational calculus as in Sections 2.3 and 3.1. The condition of fixed expectation value of energy is expressed with the help of Lagrange multiplier denoted by λ :

$$d\Sigma - \lambda_0 d \text{tr}(\mathsf{T}_{\text{int}}) - \lambda d \text{tr}(\mathsf{T}_{\text{int}} \mathsf{H}_{\text{int}}) = 0 .$$

Substituting from Eq. (3.5) for the differential of entropy, we obtain

$$\mathsf{T}_{\text{int}} = \exp(-1 - \lambda_0 - \lambda \mathsf{H}_{\text{int}}) .$$

Hence, T_{int} is a Gibbs state for some temperature T , where

$$\lambda(E) = \frac{1}{kT}$$

and k is Boltzmann constant (actually, this is a general theorem, see, e.g., [40]).

The values of λ_0 and λ are obtained from the normalisation condition, $\text{Tr}(\mathsf{T}_{\text{int}}) = 1$, and Eq. (3.73) for the expectation value of energy. The normalisation condition yields

$$\exp(1 + \lambda_0) = Z(\lambda) ,$$

where

$$Z(\lambda) = \text{Tr}(\exp(-\lambda \mathsf{H}_{\text{int}})) \quad (3.74)$$

is the partition function of the phonons. We have then

$$\frac{1}{Z(\lambda)} \frac{dZ(\lambda)}{d\lambda} = E ,$$

which gives the relation between the expectation value energy and the temperature. From the first part of Lemma 3.2, it follows that

$$\frac{1}{Z(\lambda)} \frac{d^2 Z(\lambda)}{d\lambda^2} = \text{Tr}(\mathsf{T}_{\text{int}} \mathsf{H}_{\text{int}}^2) .$$

Calculations are simplified if the variables \mathbf{u}_m and \mathbf{q}_m are transformed to the annihilation and creation operators \mathbf{a}_m and \mathbf{a}_m^\dagger of the phonons:

$$\mathbf{u}_m = \sqrt{\frac{\hbar}{2\mu\omega_m}}(\mathbf{a}_m^\dagger + \mathbf{a}_m) , \quad \mathbf{q}_m = i\sqrt{\frac{\hbar\mu\omega_m}{2}}(\mathbf{a}_m^\dagger - \mathbf{a}_m)$$

so that $[\mathbf{a}_m, \mathbf{a}_m^\dagger] = 1$. Then,

$$\mathbf{H}_{\text{int}} = \sum_{m=1}^N \hbar \omega_m \left(\mathbf{a}_m^\dagger \mathbf{a}_m + \frac{1}{2} \right) .$$

The eigenvalues of the phonon number operator $\mathbf{a}_m^\dagger \mathbf{a}_m$, the phonon occupation numbers, will be denoted by n_m . Then, the spectrum of \mathbf{H}_{int} is built from the mode frequencies by the formula

$$E_{\text{int}} = \sum_{m=1}^N n_m \hbar \omega_m . \quad (3.75)$$

Let us set

$$\mathbf{H}_{\text{int}} = \prod_{m=1}^N \otimes \mathbf{H}_m ,$$

where \mathbf{H}_m is the Hilbert space of the phonons of type m . For calculation of traces of operators \mathbf{A} of the form

$$\mathbf{A} = \sum_{m=1}^N \mathbf{A}_m ,$$

over \mathbf{H}_{int} , where \mathbf{A}_m is an operator on \mathbf{H}_m , we use the formula

$$\text{Tr}(\mathbf{A}) = \prod_{m=1}^N \text{tr}^{(m)}(\mathbf{A}_m) ,$$

where $\text{tr}^{(m)}$ is the trace over the Hilbert space of the photon of kind m .

An easy calculation gives

$$Z(\lambda) = \prod_{m=1}^N Z_m(\lambda) ,$$

where

$$Z_m(\lambda) = \frac{\exp(-\frac{1}{2}\lambda\hbar\omega_m)}{1 - \exp(-\lambda\hbar\omega_m)} .$$

Then,

$$\mathbf{T}_{\text{int}} = \prod_{m=1}^N \mathbf{T}^{(m)} , \quad (3.76)$$

where

$$\mathbf{T}^{(m)} = Z_m^{-1} \exp \left[-\lambda \hbar \omega_m \left(\mathbf{a}_m^\dagger \mathbf{a}_m + \frac{1}{2} \right) \right] .$$

As it is well-known, the internal energy has itself a very small relative variance, $\Delta E/E$, in the Gibbs state if N is large.

The diagonal matrix elements of \mathbf{u}_m between the energy eigenstates $|n_m\rangle$ that we shall need then are

$$\langle n_m | \mathbf{u}_m | n_m \rangle = 0, \quad \langle n_m | \mathbf{u}_m^2 | n_m \rangle = \frac{\hbar}{2\mu\omega_m} (2n_m + 1). \quad (3.77)$$

The length can be expressed in terms of modes \mathbf{u}_m using Eq. (3.64),

$$\mathbf{L} = N\xi + \sum_{m=0}^N (Y_N^m - Y_1^m) \mathbf{u}_m.$$

The differences on the right-hand side are non-zero only for odd values of m , and equal then to $-2Y_1^m$. We easily find, using Eqs. (3.67) and (3.68):

$$\mathbf{L} = N\xi - \sqrt{\frac{8}{N}} \sum_{m=1}^{[(N+1)/2]} (-1)^m \cos\left(\frac{2m-1}{N+1} \frac{\pi}{2}\right) \mathbf{u}_{2m-1}, \quad (3.78)$$

where $[(N+1)/2]$ is the entire part of $(N+1)/2$.

The expectation value of the length operator in the Gibbs state can then be obtained using Equations (3.77) and (3.76),

$$\langle \mathbf{L} \rangle = N\xi. \quad (3.79)$$

It is a function of objective properties N , ξ and E .

Eq. (3.78) is an important result. It shows that contributions to the length are more or less evenly distributed over all odd modes. Such a distribution leads to a very small variance of \mathbf{L} in Gibbs states. A lengthy calculation [35] using Eq. (3.77) gives for large N

$$\frac{\Delta \mathbf{L}}{\langle \mathbf{L} \rangle} \approx \frac{2\sqrt{3}}{\pi\kappa\xi\sqrt{\lambda}} \frac{1}{\sqrt{N+1}}. \quad (3.80)$$

Thus, the small relative variance for large N need not be assumed from the start and it guarantees the approximative match between the quantum and classical models of the real object. The only assumptions are values of some structural parameters and that an expectation value of energy is fixed. We have obtained even more information, viz. the internal-energy dependence of the length (in this model, the length is a structure parameter and the dependence is trivial). This is an objective relation that can be in principle tested by measurements.

Similar results can be obtained for further thermodynamic properties such as specific heat, elasticity coefficient² etc. All these quantities are well known to have

²If we extend the classical model so that it contains the elasticity coefficient, we could calculate the coefficient for an extended quantum model, in which the rod would be placed into a non-homogeneous “gravitational” field described by, say, a quadratic potential. This would again give a solvable model.

small variances in Gibbs states. The reason is that the contributions to these quantities are evenly distributed over the normal modes and the modes are mechanically and statistically independent.

The mechanical properties of the system are the centre of mass and the total momentum. The contributions to them are evenly distributed over all atoms, not modes: the bulk motion is mechanically and statistically independent of all other modes and so its variances will not be small in Gibbs states defined by a fixed expectation value of the total energy. Still, generalized statistical methods of Chapter 2 can be applied to it.

First, we assume that the real rod we are modelling cannot possess a sharp trajectory and that satisfactory models of it can be ME packets in both Newtonian and quantum mechanics. Then, according to Assumption 3.2 and Theorem 2.1, the external state of the classical model can be chosen as

$$\rho = \frac{v}{2\pi} \frac{1}{\langle \Delta X \rangle \langle \Delta P \rangle} \exp \left[-\frac{(X - \langle X \rangle)^2}{2\langle \Delta X \rangle^2} - \frac{(P - \langle P \rangle)^2}{2\langle \Delta P \rangle^2} \right] . \quad (3.81)$$

Similarly, Theorem 3.1 implies that the external state of the quantum model can be chosen as

$$T_{\text{CM}} = \frac{2}{\sqrt{\nu^2 - 1}} \exp \left(-\frac{\nu}{2} \ln \frac{\nu + 1}{\nu - 1} K' \right) , \quad (3.82)$$

where

$$K' = \frac{1}{2} \frac{(\mathbf{q} - Q)^2}{\Delta Q^2} + \frac{1}{2} \frac{(\mathbf{p} - P)^2}{\Delta P^2} .$$

The Hamiltonian for the bulk motion of both models is given by Eq. (3.70). Thus, as explained in Section 3.3, the quantum trajectory coincides with the classical one exactly. (Recall that trajectory has been defined as the time dependence of expectation values and variances.)

Hopefully, this simple rod example has sufficiently illustrated how our idea of model construction works in the case of classical properties and we can finish the comparison of classical and quantum models here.

Chapter 4

Measurement and exchange symmetry

Let systems $S^{(1)}$ and $S^{(2)}$ of different types in states $\mathsf{T}^{(1)}$ and $\mathsf{T}^{(2)}$ be considered as one composite system S . Then the state T of S is the tensor product,

$$\mathsf{T} = \mathsf{T}^{(1)} \otimes \mathsf{T}^{(2)} ,$$

see [58], p. 115. However, if the systems to be composed are of the same type then the situation is different:

Any [registration] performed on the [composite] quantum system treats all [indistinguishable] subsystems in the same way, and it is indifferent to a permutation of the labels that we attribute to the individual subsystems for computational purposes.

(Peres [58], p. 126.) Thus, the particle labels or tensor-product positions have no physical importance: they are superfluous variables similar to gauge in gauge field theories. This introduces another group into the foundation of quantum mechanics: the system-permutation group.

In the present chapter, we are going to show how the RCU interpretation of quantum mechanics can be adapted to exchange symmetry. In particular, we review the results of [33], especially on the incompleteness of registration apparatuses, on the interpretation of preparation processes and of quantum observables. Our understanding of systems, states and observables will thus reach its definitive shape.

4.1 Indistinguishable systems

Here, we collect briefly some well-known mathematics (see, e.g., Section 5-4 of [58] or Chapter 17 of [1]) in the form that will be useful for further development and

use it to state physical assumptions concerning indistinguishable particles in the framework of RCU interpretation.

We begin by an account of the action of the group of system permutations on tensor products of Hilbert spaces. Let \mathbf{S}_N be the permutation group of N objects, that is, each element g of \mathbf{S}_N is a bijective map $g : \{1, \dots, N\} \mapsto \{1, \dots, N\}$, the inverse element to g is the inverse map g^{-1} and the group product of g_1 and g_2 is defined by the composition of the maps, $(g_1 g_2)(k) = g_1(g_2(k))$, $k \in \{1, \dots, N\}$.

Given a Hilbert space \mathbf{H} , let us denote by \mathbf{H}^N the tensor product of N copies of \mathbf{H} ,

$$\mathbf{H}^N = \mathbf{H} \otimes \mathbf{H} \otimes \dots \otimes \mathbf{H} .$$

On \mathbf{H}^N , the permutation group \mathbf{S}_N acts as follows. Let $|\psi_k\rangle \in \mathbf{H}$, $k = 1, \dots, N$, be N vectors. Then

$$|\psi_1\rangle \otimes \dots \otimes |\psi_N\rangle \in \mathbf{H}^N$$

and

$$\mathbf{g}(|\psi_1\rangle \otimes \dots \otimes |\psi_N\rangle) = |\psi_{g(1)}\rangle \otimes \dots \otimes |\psi_{g(N)}\rangle . \quad (4.1)$$

\mathbf{g} is linear, preserves the inner product of \mathbf{H}^N and is, therefore, bounded and continuous. Hence, it can be extended by linearity and continuity to the whole of \mathbf{H}^N . The resulting operator on \mathbf{H}^N is denoted by the same symbol \mathbf{g} and is a unitary operator by construction. The action (4.1) thus defines a unitary representation of the group \mathbf{S}_N on \mathbf{H}^N .

Consider a particle with Hilbert space \mathbf{H}_s defined at the beginning of Section 1.4. The wave functions representing vectors of \mathbf{H}_s have the form $\psi(\vec{x}, m)$ in the Q -representation, in which the position and the third component of spin are diagonal. As it is well known, other representations exist, e.g. the P -representation, in which the momentum and the first component, say, of spin are diagonal and we have functions $\psi(\vec{p}, m)$, or a representation associated with some Hilbert-space basis $\{|n\rangle\}$ and we have functions of discrete numbers n so that $\psi(n)$ are coefficients in the expansion of the vector into the basis. We can work in any representation by replacing the arguments of the wave function by a single shorthand $\lambda = (\vec{x}, m)$, or $\lambda = (\vec{p}, m')$, or $\lambda = n$, etc. The scalar product of two vectors $|\psi\rangle$ and $|\phi\rangle$ can then be written as the Stieltjes integral over λ (see, e.g., [63], p. 19):

$$\langle \psi | \phi \rangle = \int d\lambda \psi^*(\lambda) \phi(\lambda)$$

so that

$$\int d\lambda \psi^*(\lambda) \phi(\lambda) = \sum_{m=-s}^s \int_{\mathbb{R}^3} d^3x \psi^*(\vec{x}, m) \phi(\vec{x}, m) = \sum_n \psi^*(n) \phi(n) ,$$

etc. This method not only shortens formulas but is also manifestly representation independent.

Next, consider two such particles. The wave functions that are associated with vectors of $\mathbf{H}_s \otimes \mathbf{H}_s$ have the form

$$\psi(\lambda^{(1)}, \lambda^{(2)}) ,$$

and the wave function associated with vector $|\psi_1\rangle \otimes |\psi_2\rangle$ is

$$\psi_1(\lambda^{(1)})\psi_2(\lambda^{(2)}) .$$

For a rigorous justification of writing tensor products in this way, see [63], p. 51. The only non-trivial element of group \mathbf{S}_2 , g , exchanges the two numbers 1 and 2 and its action on $|\psi_1\rangle \otimes |\psi_2\rangle$ is represented by

$$g(\psi_1(\lambda^{(1)})\psi_2(\lambda^{(2)})) = \psi_2(\lambda^{(1)})\psi_1(\lambda^{(2)}) = \psi_1(\lambda^{(2)})\psi_2(\lambda^{(1)}) .$$

The extension of this operation to the whole of $\mathbf{H}_s \otimes \mathbf{H}_s$ leads to the following operation on general wave functions:

$$g(\psi(\lambda^{(1)}, \lambda^{(2)})) = \psi(\lambda^{(2)}, \lambda^{(1)}) .$$

Hence, it exchanges the system labels.

Group \mathbf{S}_2 has just two irreducible representations: the symmetric one on the symmetric wave functions, and the alternating one on the antisymmetric wave functions (see, e.g., [78], Chapter 14). Clearly, (A) each wave function can be written as a sum of a symmetric and an antisymmetric one,

$$\psi(\lambda^{(1)}, \lambda^{(2)}) = \frac{1}{2}(\psi(\lambda^{(1)}, \lambda^{(2)}) + \psi(\lambda^{(2)}, \lambda^{(1)})) + \frac{1}{2}(\psi(\lambda^{(1)}, \lambda^{(2)}) - \psi(\lambda^{(2)}, \lambda^{(1)}))$$

and (B), each symmetric wave function, $\psi_s(\lambda^{(1)}, \lambda^{(2)})$ is orthogonal to each antisymmetric one, $\psi_a(\lambda^{(1)}, \lambda^{(2)})$,

$$\int d\lambda^{(1)} d\lambda^{(2)} \psi_s^*(\lambda^{(1)}, \lambda^{(2)})\psi_a(\lambda^{(1)}, \lambda^{(2)}) = 0 .$$

It then follows that the whole Hilbert space is an orthogonal sum of two subspaces, each containing only states transforming under one irreducible representation of group \mathbf{S}_2 .

In general, all vectors of \mathbf{H}^N that transform according to a fixed irreducible unitary representation \mathcal{R} of \mathbf{S}_N form a closed linear subspace of \mathbf{H}^N that will be denoted by $\mathbf{H}_{\mathcal{R}}^N$. The representations being unitary, the subspaces $\mathbf{H}_{\mathcal{R}}^N$ are orthogonal to each other (see, e.g., [78]). Let us denote by $\Pi_{\mathcal{R}}^N$ the orthogonal projection operator,

$$\Pi_{\mathcal{R}}^N : \mathbf{H}^N \mapsto \mathbf{H}_{\mathcal{R}}^N .$$

The index N at $\Pi_{\mathcal{R}}^N$ ought not to be confused with a power (in fact, the projections are idempotent).

The location order of a given state in a tensor product or the index at the variables $\lambda^{(k)}$ can be considered as an information about the identity of the corresponding system. As already mentioned, such information has no physical meaning and any permutation thereof is just a kind of gauge transformation¹. The formalism will be gauge invariant if only one-dimensional unitary representations of \mathbf{S}_N are allowed because only these transform vectors by a phase factor multiplication and thus do not change the corresponding states. Another motivation for this restriction is the cluster separability (see [58], p. 128). \mathbf{S}_N has exactly two one-dimensional unitary representations: the symmetric (trivial) one, $g \mapsto 1$, and the alternating one $g \mapsto \eta(g)1$ for each $g \in \mathbf{S}_N$, where $\eta(g) = 1$ for even and $\eta(g) = -1$ for odd permutation g [78]. If \mathcal{R} is the symmetric (alternating) representation we use symbol \mathbf{H}_+^N (\mathbf{H}_-^N) for $\mathbf{H}_{\mathcal{R}}^N$. Let us introduce index τ with values -1 and $+1$ (in indices, we just write $+$ or $-$) and let Π_{τ}^N be the orthogonal projection on \mathbf{H}_{τ}^N . The projection symmetrize or antisymmetrize in dependence of τ and can therefore be called “ τ -symmetrisation”. Note that the usual operation of symmetrisation or antisymmetrisation (we shall speak of τ -symmetrisation in general) on a vector $\Psi \in \mathbf{H}^N$, such as

$$|\psi\rangle \otimes |\phi\rangle \mapsto (1/2)(|\psi\rangle \otimes |\phi\rangle \pm |\phi\rangle \otimes |\psi\rangle)$$

for \mathbf{H}^2 , is nothing but $\Pi_+^N \Psi$ or $\Pi_-^N \Psi$, respectively.

An important property of the subspaces \mathbf{H}_{τ}^N is their invariance with respect to tensor products of unitary transformations. Let \mathbf{U} be a unitary transformation on \mathbf{H} , then $\mathbf{U} \otimes \mathbf{U} \otimes \dots \otimes \mathbf{U}$ is a unitary transformation on $\mathbf{H} \otimes \mathbf{H} \otimes \dots \otimes \mathbf{H}$ and each subspace \mathbf{H}_{τ}^N is invariant with respect to it. Hence, $\mathbf{U} \otimes \mathbf{U} \otimes \dots \otimes \mathbf{U}$ acts as a unitary transformation on \mathbf{H}_{τ}^N for each τ . A unitary representation of a group \mathbf{G} on \mathbf{H}_{τ}^N that is constructed in this way from a representation of \mathbf{G} on \mathbf{H} is the tensor product of representations (see e.g. [62], Section 5.17; there, tensor products are called “Kronecker products”).

Theorem 4.1 *Let \mathbf{G} be the generator of a one-parameter Lie group $g(t)$ of unitary operators on \mathbf{H} . Then, the generator $\tilde{\mathbf{G}}$ of $g(t)$ on \mathbf{H}_{τ}^N for the tensor product of the representations is given by*

$$\tilde{\mathbf{G}} = \mathbf{G} \otimes 1 \otimes \dots \otimes 1 + 1 \otimes \mathbf{G} \otimes 1 \otimes \dots \otimes 1 + \dots + 1 \otimes \dots \otimes 1 \otimes \mathbf{G} , \quad (4.2)$$

where 1 is the unit operator on \mathbf{H} .

¹A different and independent part (ignored here) of the theory of identical particles is that states of two identical systems can also be swapped in a physical process of continuous evolution, and can so entail a non-trivial phase factor at the total state (anyons, see, e.g. Ref. [79]).

The Theorem follows easily from the definition of group generators. Observe that the form of the generator is independent of whether the space is symmetric or antisymmetric.

Now, we are ready to formulate the basic assumption of standard quantum mechanics concerning identical subsystems. From relativistic quantum field theory (see, e.g., [77]), we take over the following result.

Assumption 4.1 *Let \mathcal{S}^N be a quantum system composite of N subsystems \mathcal{S} of the same type, each with Hilbert space \mathbf{H}_s . Then, the Hilbert space of \mathcal{S}^N is $(\mathbf{H}_s)_\tau^N$ with $\tau = (-1)^{2s}$. The definition of system \mathcal{S}^N is finished if a representation of group $\bar{\mathbf{G}}_\mu^+$ (see Section 1.4) on $(\mathbf{H}_s)_\tau^N$ is chosen.*

As it is well known, systems with integer spin are called *bosons* and those with half-integer spin are called *fermions*.

Assumption 4.2 *Let \mathbf{G} be the generator of subgroup $g(t)$ of $\bar{\mathbf{G}}_\mu^+$ on \mathbf{H} that does not contain time translations. Then, the generator $\tilde{\mathbf{G}}$ of $g(t)$ on \mathbf{H}_τ^N is given by Eq. (4.2). For time translation subgroup, suitable potential function can be added to the right-hand side of Eq. (4.2). The potential function must commute with all other generators of the group and be an operator on \mathbf{H}_τ^N .*

The constructed representation of $\bar{\mathbf{G}}_\mu^+$ on \mathbf{H}_τ^N for $N > 1$ is not irreducible and non-equivalent representations of it on \mathbf{H}_τ^N can differ by potential functions, that is addition terms in the time-translation operator that commute with all other generators of the group (see e.g. [43]). Hence, to define a system of N identical particles, one has also to specify its Hamiltonian.

For example, a possible Hamiltonian of three identical fermions with mass μ and charge e is

$$-\frac{\hbar^2}{2\mu}(\Delta^{(1)} + \Delta^{(2)} + \Delta^{(3)}) + V(\vec{x}^{(1)}, \vec{x}^{(2)}) + V(\vec{x}^{(3)}, \vec{x}^{(1)}) + V(\vec{x}^{(2)}, \vec{x}^{(3)}), \quad (4.3)$$

where Δ is the Laplacean and

$$V(\vec{x}^{(k)}, \vec{x}^{(l)}) = \frac{e^2}{4\pi\epsilon_0|\vec{x}^{(k)} - \vec{x}^{(l)}|} \quad (4.4)$$

is a potential function that is invariant with respect to translations, rotations and boosts. We can see that both the differential operator and the potential function are invariant with respect of all permutation of three numbers 1, 2 and 3. This is necessary and sufficient for the Hamiltonian to be an operator on \mathbf{H}_-^3

The first term in the Hamiltonian is defined by a one-particle operator $-\hbar^2/(2\mu)\Delta$ and the rest is defined by a two-particle operator $V(\vec{x}^{(1)}, \vec{x}^{(2)})$.

For states and observables, we have:

Assumption 4.3 *Possible states of system \mathcal{S}^N composite of N systems of the same type with spin s are state operators on \mathbf{H}_τ^N (elements of $\mathbf{T}(\mathbf{H}_\tau^N)$) and the observables of \mathcal{S}^N are self-adjoint operators on \mathbf{H}_τ^N , where $\tau = (-1)^{2s}$. The probabilities of registration outcomes are given by the corresponding Born rule.*

The well-known rules represented by Assumptions 4.1, 4.2 and 4.3 lead to deeper understanding of quantum systems. From TH's 1.1 and 1.3, we can conclude that, for any particle of a given type, there is a huge number of particles of the same type somewhere in the world. In the present chapter, we have learnt: If we label a system so that it is distinguished from another system of the same type, then such a label is just a mathematical tool and does not carry any physical information. Hence, a quantum system is just an auxiliary mathematical notion. A physical quantum object is defined by its type and its prepared state. This state can be “occupied” by any system of the type. We are going to make this statement clearer in Section 4.3.

4.2 Born rule

Here, we are going to study the connection between an observable and a meter that registers the observable. A new understanding of this connection can be achieved if the exchange symmetry is taken into account.

An observable is an n -tuple $\{\mathbf{O}_1, \dots, \mathbf{O}_n\}$ of commuting self-adjoint operators $\mathbf{O}_k : \mathbf{H}_s \mapsto \mathbf{H}_s$, $k = 1, \dots, n$. Let $\sigma \subset \mathbb{R}^n$ be the spectrum of $\{\mathbf{O}_1, \dots, \mathbf{O}_n\}$, $\mathcal{B}(\mathbb{R}^n)$ the set of Borel subsets of \mathbb{R}^n and let $\Pi(X)$, $X \in \mathcal{B}(\mathbb{R}^n)$, describe the spectral measure of $\{\mathbf{O}_1, \dots, \mathbf{O}_n\}$ (see e.g. [63], p. 228). In particular, $\Pi(X)$ is an orthogonal projection on \mathbf{H}_s for each $X \in \mathcal{B}(\mathbb{R}^n)$,

$$\Pi(X)^\dagger = \Pi(X) \ , \quad \Pi(X)^2 = \Pi(X) \ ,$$

and satisfies the normalization condition,

$$\Pi(\mathbb{R}^n) = 1 \ . \tag{4.5}$$

A relation between observables and registrations is expressed by the *Born rule*. According to the usual notion of it, the probability that a value of the observable within X will be obtained by registration on state $\mathbf{T} \in \mathbf{T}(\mathbf{H}_s)$ is $\text{tr}(\mathbf{T}\Pi(X))$. In practice, the Born rule means that the relative frequencies of the values obtained by many registration by the same meter \mathcal{M} on the same state \mathbf{T} must tend to the probabilities given by the Born rule if the number of the registrations increases. We are going to formulate this in the following cautious way.

Definition 4.1 Given state T and $X \in \mathcal{B}(\mathbb{R}^n)$, let us denote by $\omega[\mathcal{M}, \mathsf{T}](X)$ the frequencies of finding values of observable $\{\mathsf{O}_1, \dots, \mathsf{O}_n\}$ within X obtained from many registrations by meter \mathcal{M} on state T . Let $\omega[\mathcal{M}, \mathsf{T}](X)$ tends to $\text{tr}(\mathsf{T}\Pi(X))$ as the number of registration events increases,

$$\omega[\mathcal{M}, \mathsf{T}](X) \mapsto \text{tr}(\mathsf{T}\Pi(X)) , \quad (4.6)$$

for some states T and all $X \in \mathcal{B}(\mathbb{R}^n)$. Then we say that meter \mathcal{M} measures $\{\mathsf{O}_1, \dots, \mathsf{O}_n\}$.

We must also clarify, what are the relative frequencies:

Definition 4.2 The relative frequencies are defined by the number of registration events in a given series of registrations:

$$\omega[\mathcal{M}, \mathsf{T}](X) = \frac{N[\mathcal{M}, \mathsf{T}](X)}{N[\mathcal{M}, \mathsf{T}](\mathbb{R}^n)} .$$

where $N[\mathcal{M}, \mathsf{T}](X)$ is the number of registration events giving the registered value within X while $N[\mathcal{M}, \mathsf{T}](\mathbb{R}^n)$ is the number of all registration events.

Observe that $N[\mathcal{M}, \mathsf{T}](\mathbb{R}^n)$ need not be the number of all individual particles in state T shot at \mathcal{M} .

Definition 4.1 differs from the common version by a weaker requirement on the states: the frequency agrees with the Born rule on “some states” but not necessarily on “all states”. Indeed, “all states” seems to be the understanding by various books, such as [51, 58] at least implicitly, and [9] quite explicitly (“probability reproducibility condition”, p. 29). To explain Definition 4.1, let us consider some examples.

1. The position \vec{x} of particle S with Hilbert space \mathbf{H}_s is registered by some detector with active volume D (see, e.g., [49]). If the detector gives a response (clicks) then we conclude that a particle has been detected inside D . If the wave function of the detected particle is $\psi(\vec{x})$ then the probability that the particle will be found inside the detector is

$$P(D) = \int_{\mathbb{R}^3} d^3x \chi_D(\vec{x}) |\psi(\vec{x})|^2 ,$$

where $\chi_D(\vec{x})$ is the characteristic function of D . Hence, $P(D) = 0$ if $\text{supp } \psi(\vec{x}) \cap D = \emptyset$. The integral on the right-hand side represents the trace (4.6) with $\mathsf{T} = |\psi\rangle\langle\psi|$ and $\Pi(X) = \xi_X(\vec{x})$.

A real meter \mathcal{M} registering position is composed of several such sub-detectors, $\mathcal{M}_1, \dots, \mathcal{M}_n$ with disjoint active volumes D_1, \dots, D_n and the frequency of finding the particle inside D_k then satisfies

$$\omega[\mathcal{M}_k, \psi] \mapsto \int_{\mathbb{R}^3} d^3x \chi_{D_k}(\vec{x}) |\psi(\vec{x})|^2 .$$

Such detector gives some information only about the subset of $\mathcal{B}(\mathbb{R}^n)$ generated from $\{D_1, \dots, D_n\}$ by set unions. The meter does not register the whole spectrum but only the part $\sigma'_x \subset \sigma_x$ defined by

$$\sigma'_x = \bigcup_{k=1}^n D_k .$$

Hence, for all states ψ such that

$$\text{supp } \psi(\vec{x}) \subset \sigma'_x \tag{4.7}$$

the detector satisfies the Born rule for all $X \in \mathcal{B}(\mathbb{R}^3)$ because zero probability for S being outside of σ'_x results from both the Born rule and the registrations by \mathcal{M} . However, for the states that do not satisfy Eq. (4.7), the meter still gives zero probability for S being outside of σ'_x contradicting the Born rule.

The meter does not react at all to any wave function satisfying

$$\text{supp } \psi(\vec{x}) \subset \mathbb{R}^3 \setminus \bigcup_{k=1}^n D_k .$$

Not to react to state ψ means that

$$\omega[\mathcal{M}_k, \psi] = 0$$

for all k .

2. Next, consider an electron and its energy. A meter that can register energy is, e.g., a proportional counter. Such a meter reacts to a free particle if its energy is larger than some non-zero threshold. Moreover, if the electron is bound, as in a hydrogen atom, then it is impossible to register its energy directly, that is by a meter that reacts to the state of electron as a subsystem of the atom. This is different from registrations of the energy of the atom itself. All energy levels of its bounded states can be registered indirectly (by scattering of photons, say), or directly (at least in principle, see [56]). On top of that, the spectrum of free electron is continuous, and no meter can distinguish any two eigenvalues, if they are too close to each other. Hence, it does not cover

the whole spectrum but only a finite number of energy intervals of a sufficient width.

To summarize, such a meter gives information only about some part of $\mathcal{B}(\mathcal{R})$ and it will not react to a number of states. This example shows that the problem need not be caused just by the geometric arrangement of the experiment as in point 1.

3. The Stern-Gerlach apparatus as described in Section 1.3 can register the spin observable only if the particle arriving at it can pass through the opening between the magnets. Although it gives information about the whole of $\mathcal{B}(\mathcal{R})$, it will not react to a large set of states. This example shows that the problem can arise even for a meter that registers the whole spectrum.

Let us compare this with the well-known cases (see, e.g., [9]) of meters that do not register the whole spectrum. For instance, a real meter can only discriminate between sufficiently different values of an observable \mathbf{O} with a continuous spectrum that is, it registers only some coarse-grained version of the spectrum. Thus, one introduces a finite partition the space \mathbb{R}^n ,

$$\mathbb{R}^n = \bigcup_{l=1}^n X_l ,$$

and considers only projections $\Pi(X_l)$, $l = 1, 2, \dots, n$, which define a new observable with spectrum $\{1, 2, \dots, n\}$ (for details, see [9], p. 35). Notice that the idea is to modify the observable so that the correspondence between observable and meter via the Born rule is improved. The meter then does react to all states of the system and satisfies the Born rule corresponding to the corrected observable. Still, the method does not work for the cases above, in which the Borel sets that are controlled by the apparatus do not cover the spectrum. Such an apparatus does not react to states corresponding to the part of the spectrum that is not covered, so that the difficulty with the states also occurs.

The possibility that a meter may control only a (sometimes rather small) proper subset of the whole Hilbert space, as the apparatuses of the above examples do, does not seem to be even mentioned in the literature. This might be due to the belief that, as in most cases of non-ideal real circumstances, the shortcoming of real apparatuses is a natural way of practical things which just must be taken into account if necessary in each particular instance and that some real apparatuses might be arbitrarily close to the ideal or, at least, that continuous improvement of techniques will make apparatuses ever better. The main aim of the present chapter is to show that quantum mechanics of indistinguishable particles sets an objective limit to this: an apparatus that were ideal in this sense could not register its observable at all.

4.3 Incompleteness of registration apparatuses

Let us now simplify things by considering observables described by a single operator ($n = 1$). The foregoing section motivates the following definition.

Definition 4.3 *Let S be a quantum system with Hilbert space \mathbf{H} and let observable \mathbf{O} be a s.a. operator on \mathbf{H} with spectrum σ and spectral measure $\Pi(X)$, $X \in \mathcal{B}(\mathbb{R})$. Let meter \mathcal{M} register observable \mathbf{O} . We say that \mathcal{M} is complete if Eq. (4.6) holds true for all $\mathbf{T} \in \mathbf{T}(\mathbf{H})$ and $X \in \mathcal{B}(\mathbb{R})$.*

If there is any state \mathbf{T} for which the frequency $\omega[\mathcal{M}, \mathbf{T}](\mathbb{R})$ of registering any value by \mathcal{M} is zero, meter \mathcal{M} is called incomplete. Let the subset of states for which this is the case be denoted by $\mathbf{T}(\mathbf{H})_{\mathcal{M}0}$ and the subset of states for which Eq. (4.6) holds by $\mathbf{T}(\mathbf{H})_{\mathcal{M}}$. The convex set $\mathbf{T}(\mathbf{H})_{\mathcal{M}}$ is called the domain of \mathcal{M} .

Thus, the three examples in the foregoing section describe incomplete apparatuses. We are now going to prove that a complete meter cannot work.

Let registrations by meter \mathcal{M} be performed on a system S with Hilbert space \mathbf{H} . Suppose that meter \mathcal{M} registers observable \mathbf{O} with spectral measure $\Pi(X)$, $X \in \mathcal{B}(\mathbb{R})$, and is complete. Then, because of the normalisation condition (4.5), we must have $\text{tr}(\mathbf{T}\Pi(\mathbb{R})) = 1$ for *any* state \mathbf{T} . This means that *any* registration on \mathbf{T} by \mathcal{M} must give *some* result.

Then, according to the theory of indistinguishable systems, \mathcal{M} must also register some values on any state \mathbf{T}' of any system S' of the same type as S . Clearly, this is a difficulty: the measurement of observable \mathbf{O} of S by \mathcal{M} is disturbed by the existence of a system of the same type as S anywhere else in the world, even if it is localised arbitrarily far away from S because it cannot be distinguished from S by \mathcal{M} . According to Trial Hypotheses 1.1 and 1.3, for most microsystems S , the world contains a huge number of systems of the same type so that a horrible noise must disturb any registration by a complete meter.

To show the problem in more detail, let us consider two distant laboratories, A and B. Let \mathbf{O} be a non-degenerate discrete observable of S with eigenstates $|k\rangle$ and eigenvalues o_k . Let state $|k\rangle$ be prepared in A and $|l\rangle$ in B so that $k \neq l$ and let \mathbf{O} be registered in laboratory A by complete meter \mathcal{M} . Using Fock space formalism (see e.g. [58], Section 5-6), we have

$$\mathbf{O} = \sum_n o_n \mathbf{a}_n^\dagger \mathbf{a}_n , \quad (4.8)$$

where \mathbf{a}_k is an annihilation operator of state $|k\rangle$. Such an observable perfectly expresses the fact that the meter cannot distinguish particles of the same type. The state prepared by the two laboratories is

$$\mathbf{a}_k^\dagger \mathbf{a}_l^\dagger |0\rangle . \quad (4.9)$$

For the average $\langle O \rangle$ of (4.8) in state (4.9), the standard theory of measurement gives

$$\langle O \rangle = \langle 0 | a_l a_k \left(\sum_n a_n a_n^\dagger a_n \right) a_k^\dagger a_l^\dagger | 0 \rangle .$$

Using the relation

$$a_r a_s^\dagger = \eta a_s^\dagger a_r + \delta_{rs} ,$$

where $\eta = 1$ for bosons and $\eta = -1$ for fermions, we can bring all annihilation operators to the right and all creation ones to the left obtaining

$$\langle O \rangle = a_k + a_l .$$

The result is independent of the distance between the laboratories. Thus, the measurement in A by any complete meter depends on what is done in B.

Let us next suppose that \mathcal{M} is incomplete in such a way that the state of any system of the same type as S that may occur in the environment of S lies in $\mathbf{T}(\mathbf{H})_{\mathcal{M}0}$. Apparently, such an assumption can be checked experimentally by looking at the level of noise of the meter. Then, if we prepare a copy of system S in a state that lies within $\mathbf{T}(\mathbf{H})_{\mathcal{M}}$ the registration of S by \mathcal{M} cannot be disturbed by the systems in the environment. In fact, this must be the way of how all quantum measurement are carried out. We can say that objective properties of our environment require certain kind of incompleteness of registration apparatus \mathcal{M} in order that \mathcal{M} can work in this environment.

Accordingly, the course of any successful measurement must be as follows. First, a registration apparatus \mathcal{M} for a system S with Hilbert space \mathbf{H} is constructed and checked. In particular, the level of its noise must be sufficiently low. From the construction of the meter, we can infer some set $\mathbf{T}_{\mathcal{M}}$ of states on which the meter is able make registrations. $\mathbf{T}_{\mathcal{M}}$ might be smaller than the whole domain,

$$\mathbf{T}_{\mathcal{M}} \subset \mathbf{T}(\mathbf{H})_{\mathcal{M}}$$

(the domain is often difficult to specify). Second, systems S is prepared in one of such states. The registration by \mathcal{M} will then not be disturbed and the probabilities of the results can be calculated by formula (4.6).

The states of $\mathbf{T}_{\mathcal{M}}$ must therefore be in some sense sufficiently different from the states of all systems of the same type as S that occur in the environment of S . Let us try to express this idea mathematically. This can be done in the simplest way, if we choose a particular representation so that the wave function of an extremal state $|\psi\rangle$ will be $\psi(\lambda)$ and the kernel of a state of arbitrary external object will be $T(\lambda^{(1)}, \dots, \lambda^{(N)}; \lambda^{(1)'}, \dots, \lambda^{(N)'})$.

Then we propose the following definition:

Definition 4.4 *Let system S with Hilbert space \mathbf{H} be prepared in state $|\psi\rangle\langle\psi| \in \mathbf{T}(\mathbf{H})$. Let the environment consists of macroscopic objects with quantum models and well-defined quantum states. Let O be the quantum model associated with such an object, O_S the subsystem of O containing all subsystems of O that are indistinguishable from S and let \mathbf{T} be the state of O_S . If*

$$\int d\lambda^{(1)'} T(\lambda^{(1)}, \dots, \lambda^{(N)}; \lambda^{(1)'}, \dots, \lambda^{(N)'}) \psi(\lambda^{(1)'}) = 0 \quad (4.10)$$

holds for any object in the environment, then $|\psi\rangle\langle\psi|$ is said to have separation status.

The definition can easily be extended to states of S that are not extremal. Some motivation of the definition is as follows. Suppose that there is a system in the environment in a state ϕ and that $\langle\phi|\psi\rangle \neq 0$. Then,

$$\phi = c_1\psi + c_2\psi^\perp,$$

with non-zero c_1 and $\langle\psi^\perp|\psi\rangle = 0$, and the meter would react to the ψ -part of ϕ .

The above ideas also have some relevance to the meaning of preparation processes in quantum measurements. Stating such meaning extends the Minimum Interpretation, for which preparations and registrations are primitive notions (see, e.g., [58], p. 12).

Trial Hypothesis 4.1 *Any preparation of a single microsystem must yield a state having a separation status.*

A separation status of a microsystem is a property that is uniquely determined by a preparation. Hence, it belongs to objective properties of quantum systems (Section 1.2). But it is a property that is a necessary condition for any other objective property because each preparation must create a separation status. Moreover, only a separation status makes a quantum system distinguishable from each other system in the environment except for those that are used in the experiment being performed. Thus, a quantum object can come into being, namely in a preparation process, and can expire, viz. if it loses its separation status.

In this connection, the question of a separation status of a macroscopic body can arise. It seems that any macroscopic body in our environment is distinguishable, as a quantum system, from any other quantum system just because of its composition. It is very implausible that there can be, somewhere, a macroscopic system that is identical, in the quantum mechanical sense, to a given macroscopic body.

We can understand the role of incompleteness of meters better if we compare quantum apparatuses with classical ones. To this aim, we construct a simple model of an eye. Indeed, an eye is a classical registration device, either by itself or as a final part of other classical apparatuses.

Our model contains of an optically sensitive surface (retina) that can register visible light (i.e., with a wave-length between 0.4 and $0.75 \mu\text{m}$). It can distinguish between some small intervals of the visible wave lengths and between small spots where the retina is hit by photons.

The retina covers one side of a chamber that has walls keeping light away except for a small opening at the side opposite to the retina wall². The radius of the circular opening, even if very small, is much larger than the wave length of visible light so that this light waves suffer only a negligible bending as they pass the opening. Hence, the assumed insensitivity of the retina to smaller wave lengths is an incompleteness that helps to make the picture sharp.

Another aspect of incompleteness is that only the light that can pass the opening will be registered. Again, this is important: if the retina were exposed to all light that can reach it from the neighbourhood, only a smeared, more or less homogeneous signal would result. Because of the restrictions, a well-structured colour picture of the world in front of the eye will appear on the retina.

Clearly classical apparatuses must also be incomplete in order to yield a non-trivial information. Of course, they give information about value distributions obtained simultaneously for a great number of particles rather than an information on a single particle.

4.4 Observables

There are two ways of how one could react to the necessary incompleteness of registration apparatuses. First, one can try to modify the definition of the observable that is registered by such a meter so that the results of the registrations and the probabilities calculated from the Born rule coincide, similarly as it has been done above for coarse-grained version of the spectrum. Second, one can leave the observables as they are and accept the fact that each meter can register its observable only partially as in Definition 4.3. In our previous work ([26, 27, 36]), we have tried the first way. It turned out, however (see [33]), that the modification that was necessary for an observable to describe how a real meter worked was messy. Not only the notion of observable became rather complicated but also just some idealized kinds of meters could be captured in this way.

The mentioned idealized kind of incomplete meter can be described as follows. Such a meter \mathcal{M} determines a closed linear subspace \mathbf{H}_{ss} of \mathbf{H} so that, instead of Eq. (4.6), we have

$$\omega[\mathcal{M}, \mathbf{T}](X) \mapsto \text{tr} \left(\left(\Pi_{\text{ss}} \mathbf{T} \Pi_{\text{ss}} \right) \Pi(X) \right)$$

²An eye with a small opening instead of a lens occurs in some animals such as nautilus.

for all $\mathsf{T} \in \mathbf{T}(\mathbf{H})$ and $X \in \mathcal{B}(\mathbb{R})$, where Π_{ss} is the orthogonal projection onto \mathbf{H}_{ss} . Then,

$$\mathbf{T}(\mathbf{H})_{\mathcal{M}} = \mathbf{T}(\mathbf{H}_{\text{ss}}) \quad (4.11)$$

and

$$\mathbf{T}(\mathbf{H})_{\mathcal{M}0} = \mathbf{T}([1 - \Pi_{\text{ss}}]\mathbf{H})$$

because any element T of $\mathbf{T}(\mathbf{H}_{\text{ss}})$ satisfies

$$\mathsf{T} = \Pi_{\text{ss}} \mathsf{T} \Pi_{\text{ss}} . \quad (4.12)$$

The construction of the corresponding modified observable can go as follows. First, we have

$$\text{tr} \left(\left(\Pi_{\text{ss}} \mathsf{T} \Pi_{\text{ss}} \right) \Pi(X) \right) = \text{tr} \left(\mathsf{T} \left(\Pi_{\text{ss}} \Pi(X) \Pi_{\text{ss}} \right) \right) .$$

Next, consider operator $\Pi_{\text{ss}} \Pi(X) \Pi_{\text{ss}}$. It is bounded by 1 and self adjoint because Π_{ss} and $\Pi(X)$ are. It is obviously positive. Thus, it is an *effect* (a positive operator bounded by 1, see [11]). A collection of effects $\mathsf{E}(X)$, $X \in \mathcal{B}(\mathbb{R})$, with certain properties (including the normalisation condition $\mathsf{E}(\mathbb{R}) = 1$) is called a *positive-operator valued measure* (POVM) (see [11] or [9]) and generalizes the notion of spectral measure. The collection of effects $\Pi_{\text{ss}} \Pi(X) \Pi_{\text{ss}}$ for all $X \in \mathcal{B}(\mathbb{R})$ is not a POVM, however, because we have, instead of the above normalisation condition,

$$\Pi_{\text{ss}} \Pi(\mathbb{R}) \Pi_{\text{ss}} = \Pi_{\text{ss}}$$

Such a quantity could be called “truncated POVM”. Thus, the notion of observable had to be changed from a self-adjoint operator to a truncated POVM.

However, the above model of incomplete meter is too simple. For instance, some of the examples listed in Section 4.2 cannot be described by it. Indeed, consider the Stern-Gerlach meter that is arranged in such a way that it can react to particles moving within a thin tube around the third axis of coordinates x_1, x_2, x_3 . The particle that can be registered must thus arrive at the magnets only within some small subset of the (1,2)-plane, the third component of its momentum must satisfy

$$p_3 \in (a_3, b_3) ,$$

which must be large enough to lie above the detector threshold, and

$$p_1 \in (-c_1, c_1) , \quad p_2 \in (-c_2, c_2) ,$$

where $c_k < \epsilon$ for $k = 1, 2$ and for sufficiently small ϵ . However, these conditions can be satisfied, by any wave packets, only approximately. Then, the Born rule will

also be satisfied only approximately. Now, a linear superposition of such packets need not be again such a packet. The above conditions mean that the wave function (in Q - or P -representation) of the registered particle must satisfy inequalities of the form

$$|\psi(\lambda)|^2 < \epsilon'$$

for some fixed values of λ determined by the arrangement, where λ stands either for \vec{x} or for \vec{p} , and ϵ' is a small positive number. Suppose that another wave function, ϕ , also satisfies the condition. Then it only follows, for all c and c' satisfying $|c|^2 + |c'|^2 = 1$, that

$$|c\psi(\lambda) + c'\phi(\lambda)|^2 < 2\epsilon'.$$

Hence, the packets need not form a closed linear subspace of \mathbf{H} .

These problems do not afflict the second way because the approach using incomplete meters works even if the domain of a meter does not satisfy Eq. (4.11). In fact, the knowledge of the whole domain $\mathbf{T}(\mathbf{H})_{\mathcal{M}}$ of a meter is not necessary for the construction of a model of a registration by \mathcal{M} because it is sufficient to know only those elements of $\mathbf{T}(\mathbf{H})_{\mathcal{M}}$ that are prepared for the experiment.

These are the reasons why we adopt the second way in the present review. Then, the Minimum Interpretation of observables, as represented e.g. by Assumption 4.3, has the following refinement.

Trial Hypothesis 4.2 *For any observable \mathbf{O} of a system \mathcal{S} , there is a meter that can register \mathbf{O} . There is no meter that can register \mathbf{O} on any state.*

There might be cases in which there is a meter $\mathcal{M}_{\mathbf{T}}$ registering \mathbf{O} on \mathbf{T} for every given state \mathbf{T} . Still, as each of these meters must be incomplete, a non-trivial set of meters would then be needed for a complete registration of \mathbf{O} .

Everything that has been said in this and the foregoing sections can easily be extended if the notion of observable is generalized from a self-adjoint operator to a POVM as in [51, 11, 9]. Indeed, POVM's can be considered as a generalization of observables: an observable is uniquely determined by its spectral decomposition and a spectral decomposition is a special case of POVM. Thus, some physicists started to define the notion of observable as POVM. However, there are important differences between the applications of the two notions: the observables are more handy for purposes that cannot be helped by POVM's (for instance, basing quantum mechanics on normed algebras of bounded observables— C^* -algebras [24]), or the relation of observables to the spacetime transformations) while POVM's are more closely related to particular registration methods. In this book, we keep the old notion of observables (as self-adjoint operators) and use POVM's where it is advantageous in connection with registrations.

4.5 Tensor-product method

We have seen in the foregoing sections that the disturbance of measurement by environmental particles can be avoided, if the meter is incomplete and the measured system is prepared in a state with a separation status.

The present section is going to study the resulting mathematics of quantum measurement theory in more detail. In particular, we shall consider two *ways of description* of composite-system states. The first way works with the tensor product of the environmental and the registered system states and the second one with the τ -symmetrized state of the whole composite system as required by rules of the theory of indistinguishable systems. On the one hand, the second way of description is in any case the correct one so that we have to show that the two descriptions lead to the same measurable results. On the other hand, the first way is the only practically feasible one because it does not require the knowledge of the environment state.

To develop the two descriptions, let us consider system S and its environment E with the system E_S of all its subsystems that are indistinguishable from S . Let \mathbf{H} be the Hilbert space of S and let E_S consist of N subsystems so that the Hilbert space of E_S is \mathbf{H}_τ^N . For the sake of simplicity, we assume that the states of S and E_S are extremal. The proof for general states is analogous. Let $\psi(\lambda)$ be the wave function of S . The wave function of E_S has the form

$$\Psi(\lambda^{(1)}, \dots, \lambda^{(N)}) \in \mathbf{H}_\tau^N .$$

Then the wave functions of the two descriptions are

$$\Psi(\lambda^{(1)}, \dots, \lambda^{(N)})\psi(\lambda^{(N+1)}) \quad (4.13)$$

and

$$N_{\text{exch}} \Pi_\tau^{N+1} \left(\Psi(\lambda^{(1)}, \dots, \lambda^{(N)})\psi(\lambda^{(N+1)}) \right) , \quad (4.14)$$

where $\Pi_\tau^{N+1} : \mathbf{H}_\tau^N \otimes \mathbf{H}_\tau \mapsto \mathbf{H}_\tau^{N+1}$ is an τ -symmetrisation defined in Section 4.1. Orthogonal projections do not preserve normalization. Hence, the projection must be followed by a normalization factor, which we will denote by N_{exch} standing before the projection symbol. Of course, N_{exch} depends on the projection and the wave function being projected, but we just write N_{exch} instead of $N(\Pi_\tau^N, \Psi\psi)$ to keep equations short.

As Ψ is already τ -symmetric and normalised, the expression (4.14) can be rewritten as follows:

$$\begin{aligned} N_{\text{exch}} \Pi_\tau^{N+1} \left(\Psi(\lambda^{(1)}, \dots, \lambda^{(N)})\psi(\lambda^{(N+1)}) \right) \\ = N' \sum_{K=1}^{N+1} (\tau)^{N+1-K} \Psi(\lambda^{(K+1)}, \dots, \lambda^{(N+1)}, \lambda^{(1)}, \dots, \lambda^{(K-1)})\psi(\lambda^{(K)}) , \end{aligned} \quad (4.15)$$

where $N' \neq N_{\text{exch}}$ is a suitable normalisation factor. This relation will simplify some subsequent calculations.

Eq. (4.14) shows that we can recover the second description from the first one, but if the two descriptions are to be equivalent, the first description must be derivable from the second one, too. For this aim, the separation status is necessary. Let state $\psi(\lambda)$ be prepared with separation status and let $\Pi_\psi = |\psi\rangle\langle\psi|$. Eq. (4.10) then implies

$$N' = \frac{1}{\sqrt{N+1}} .$$

Now, we make use the of fact that the operators on \mathbf{H} can act on different wave functions (elements of \mathbf{H}) in a product and that this action can be specified by the argument of the function. For example, if we have product $\phi(\lambda^{(1)})\phi'(\lambda^{(2)})$ and operator $\mathbf{O} : \mathbf{H} \mapsto \mathbf{H}$, operator $\mathbf{O}^{(1)} : \mathbf{H} \otimes \mathbf{H} \mapsto \mathbf{H} \otimes \mathbf{H}$ is defined by

$$\mathbf{O}^{(1)} \left[\phi(\lambda^{(1)})\phi'(\lambda^{(2)}) \right] = (\mathbf{O}\phi)(\lambda^{(1)})\phi'(\lambda^{(2)})$$

while $\mathbf{O}^{(2)}$ by

$$\mathbf{O}^{(2)} \left[\phi(\lambda^{(1)})\phi'(\lambda^{(2)}) \right] = \phi(\lambda^{(1)})(\mathbf{O}\phi')(\lambda^{(2)}) .$$

From the definition of separation status we then obtain that

$$\Pi_\psi^{(k)} \Psi((\lambda^{(1)}), \dots, (\lambda^{(N)})) = 0$$

for any $k = 1, \dots, N$.

With this notation, we can achieve our aim: Eq. (4.15) implies that

$$\begin{aligned} \Pi_\psi^{(N+1)} \left(N_{\text{exch}} \Pi_\tau^{N+1} \left(\Psi(\lambda^{(1)}, \dots, \lambda^{(N)}) \psi(\lambda^{(N+1)}) \right) \right) \\ = \nu_\psi \Psi(\lambda^{(1)}, \dots, \lambda^{(N)}) \psi(\lambda^{(N+1)}) , \end{aligned} \quad (4.16)$$

where ν_ψ is a suitable normalisation factor. Observe that this operation is naturally described by tensor products of τ -symmetrized wave functions rather than by the Fock-space formalism. The exchange symmetry is not violated because we can use $\Pi_\psi^{(K)}$ for any fixed $K = 1, \dots, N$ instead of $\Pi_\psi^{(N+1)}$ and the result will again be the above wave function with rearranged arguments.

The next point is to give an account of registration by an incomplete meter. We construct two observables that represent the meter, each for one of the two description ways, and show that the two ways lead to the same results. We work with a simple model to show the essential points; the general situation can be dealt with in an analogous way.

Let meter \mathcal{M} register observable $\mathbf{O} : \mathbf{H} \mapsto \mathbf{H}$ that is additive, discrete and non-degenerate. Let its eigenvalues be o_k and eigenfunctions be $\psi_k(\lambda)$, $k \in \mathbb{N}$. Let

\mathcal{M} be incomplete in the way that it reacts only to ψ_k if $k = 1, \dots, K$ for some $K \in \mathbb{N}$. Hence, the subspace \mathbf{H}_{ss} is spanned by vectors $\psi_k(\lambda)$, $k = 1, \dots, K$, and the projection onto it is

$$\Pi_{\text{ss}} = \sum_{k=1}^K \Pi_k ,$$

where

$$\Pi_k = |\psi_k\rangle\langle\psi_k| .$$

The action of the meter can now be described as follows. Let us prepare state ψ with a separation status. Hence, $\psi \in \mathbf{H}_{\text{ss}}$ and its decomposition into the eigenstates of \mathbf{O} is

$$\psi = \sum_{k=1}^K c_k \psi_k$$

with $\sum_{k=1}^K |c_k|^2 = 1$. Then the probability P_k of registering o_k on ψ is

$$P_k = \langle\psi|\Pi_k|\psi\rangle^2 ,$$

or

$$P_k = |c_k|^2$$

for $k \leq K$ and $P_k = 0$ for $k > K$.

Let us start with the first way, Eq. (4.13). We define the corresponding observable by restricting the action of \mathbf{O} or Π_k to the second factor:

$$\begin{aligned} (1 \otimes \Pi_k) \left[\Psi(\lambda^{(1)}, \dots, \lambda^{(N)}) \otimes \psi(\lambda^{(N+1)}) \right] &\equiv \Pi_k^{(N+1)} \left[\Psi(\lambda^{(1)}, \dots, \lambda^{(N)}) \psi(\lambda^{(N+1)}) \right] \\ &= c_k \Psi(\lambda^{(1)}, \dots, \lambda^{(N)}) \psi_k(\lambda^{(N+1)}) \end{aligned} \quad (4.17)$$

for $k \leq K$. Eq. (4.17) specifies the Born rule of the observable. Now, coming to the second way of description, Eq. (4.14), we use the fact that the observable is additive. For example, it acts on product $\phi(\lambda^{(1)})\phi'(\lambda^{(2)})$ as follows

$$(\mathbf{O}^{(1)} + \mathbf{O}^{(2)}) \left(\phi(\lambda^{(1)})\phi'(\lambda^{(2)}) \right) .$$

Then, to define the observable registered by \mathcal{M} , we need the action of its projection Π'_k for eigenvalue o_k , $k \in \mathbb{N}$. Let us try:

$$\Pi'_k = \sum_{l=1}^{N+1} (\Pi_k \Pi_{\text{ss}})^{(l)} . \quad (4.18)$$

Observe that operators Π_k and Π_{ss} commute. Then, using Eq. (4.15), we obtain

$$\begin{aligned}
\sum_{l=1}^{N+1} (\Pi_k \Pi_{ss})^{(l)} \left[N_{\text{exch}} \Pi_{\tau}^{N+1} \left(\Psi(\lambda^{(1)}, \dots, \lambda^{(N)}) \psi(\lambda^{(N+1)}) \right) \right] &= \sum_{l=1}^{N+1} (\Pi_k \Pi_{ss})^{(l)} \\
&\left[\frac{1}{\sqrt{N+1}} \sum_{K=1}^{N+1} (\tau)^{N+1-K} \Psi(\lambda^{(K+1)}, \dots, \lambda^{(N+1)}, \lambda^{(1)}, \dots, \lambda^{(K-1)}) \psi(\lambda^{(K)}) \right] \\
&= \frac{1}{\sqrt{N+1}} \sum_{K=1}^{N+1} (\tau)^{N+1-K} \Psi(\lambda^{(K+1)}, \dots, \lambda^{(N+1)}, \lambda^{(1)}, \dots, \lambda^{(K-1)}) c_k \psi_k(\lambda^{(K)}) \\
&= c_k N_{\text{exch}} \Pi_{\tau}^{N+1} \left(\Psi(\lambda^{(1)}, \dots, \lambda^{(N)}) \psi_k(\lambda^{(N+1)}) \right)
\end{aligned}$$

because operator $(\Pi_k \Pi_{ss})^{(l)}$ annihilates the state to the right if the argument $\lambda^{(l)}$ is in function Ψ and gives $c_k \psi_k(\lambda^{(l)})$ if the argument is in ψ . Thus, the Born rules for the observables of the two ways of description coincide.

In general, operator (4.18) is not a projection because the product

$$(\Pi_k \Pi_{ss})^{(m)} (\Pi_k \Pi_{ss})^{(n)}$$

does not in general vanish for $m \neq n$ and then $(\Pi'_k)^2 \neq \Pi'_k$. However, on the subspace of $\Pi_{\tau}^{N+1}(\mathbf{H}_{\tau}^N \otimes \mathbf{H}_{ss})$ with which we are working, the product is non-zero only if $m = n$, so that it is a projection under these conditions.

The last question is whether the dynamical evolutions for the two ways of description are compatible. First, we define the corresponding Hamiltonians. Let $\mathbf{H} : \mathbf{H}_{\tau}^N \otimes \mathbf{H} \mapsto \mathbf{H}_{\tau}^N \otimes \mathbf{H}$ be a Hamiltonian for the first way of description and let us assume that

$$\mathbf{H} \Pi_{\tau}^{N+1} = \Pi_{\tau}^{N+1} \mathbf{H}.$$

Such a Hamiltonian leaves the subspace $\Pi_{\tau}^{N+1}(\mathbf{H}_{\tau}^N \otimes \mathbf{H})$ invariant and can also be viewed as a Hamiltonian for the second way of description. Then, the two Schrödinger equations that we are going to compare are:

$$\mathbf{H}[\Psi(\lambda^{(1)}, \dots, \lambda^{(N)}) \psi(\lambda^{(N+1)})] = i\hbar \frac{\partial}{\partial t} [\Psi(\lambda^{(1)}, \dots, \lambda^{(N)}) \psi(\lambda^{(N+1)})] \quad (4.19)$$

for the first way of description and

$$\mathbf{H} \Pi_{\tau}^{N+1} [\Psi(\lambda^{(1)}, \dots, \lambda^{(N)}) \psi(\lambda^{(N+1)})] = i\hbar \frac{\partial}{\partial t} \Pi_{\tau}^{N+1} [\Psi(\lambda^{(1)}, \dots, \lambda^{(N)}) \psi(\lambda^{(N+1)})] \quad (4.20)$$

for the second way (the normalization factors cancel).

The compatibility can only be proved if the evolution preserves the separation status. Mathematically, this means that the Hamiltonian must commute with the projections defining the status:

$$\mathbf{H} \Pi_{ss}^{(k)} = \Pi_{ss}^{(k)} \mathbf{H} \quad (4.21)$$

for all $k = 1, \dots, N+1$. Then, the projections are conserved and their eigenspaces are stationary. In the case under study, this implies that the time derivative commutes with the projections, too:

$$\frac{\partial}{\partial t} \Pi_{\text{ss}}^{(k)} = \Pi_{\text{ss}}^{(k)} \frac{\partial}{\partial t} \quad (4.22)$$

for all $k = 1, \dots, N+1$.

Now, the proof of the compatibility is very simple: applying projection $\Pi_{\text{ss}}^{(N+1)}$ to both sides of Eq. (4.20) and using Eqs. (4.16) (where $\Pi_{\psi}^{(N+1)}$ can be replaced by $\Pi_{\text{ss}}^{(N+1)}$), (4.21) and (4.22), we obtain Eq. (4.19).

For processes, in which e.g. a measured system loses its separation status, the two evolutions are not compatible and the second way equation must be used. Such processes occur during registration. Many examples of such registrations have been given in [28, 31], but the separation was defined in a different way. We shall adapt the examples to the present definition of separation status in Chapter 5.

Chapter 5

Taking detectors seriously

This chapter has two aims. First, it reformulates the standard quantum theory of measurement so that it becomes compatible with the ideas of Chapters 3 and 4. Second, it states three trial hypotheses about the course of state reduction that make the theory more specific and definite than the standard one is. In particular, it postulates objective conditions under which the state reduction occurs. The chapter gives an account of the theory that has gradually evolved from some ideas of references [27, 26, 28] and were described in [34].

The Trial Hypotheses of Section 1.2 associate quantum states with individual quantum systems. As it is well known, this idea has an undesired lateral effect: the application of Schrödinger equation to the process of measurement leads to results that contradict some evidence, see [1], Section 9.2 and [58], p. 374. This motivated the introduction of a new phenomenon, the so-called *state reduction*: a non-linear correction to Schrödinger equation.

Let us first briefly recapitulate some current ideas concerning the state reduction.

5.1 Standard theory of measurement

Here, we give a short review of the theory of measurement as it is employed in the analysis of many measurements today and as it is described in, e.g., [80, 7, 71]—this is what we call “standard theory of measurement”.

The standard theory considers two systems: *object system*¹ S with Hilbert space \mathbf{H} on which the measurement is done, and *meter* or *apparatus* M_q that performs the registration. Quantum states refer to individual quantum systems. A measurement process is split into three steps.

¹“object system” is a notion of the standard theory of measurement which has nothing to do with our notions of “object” and “system”

1. Initially, quantum system S is prepared in state T_i^S and the meter M_q is prepared in state T_i^M . The preparations of the object system and the meter are assumed to be independent from each other so that the composite $S + M_q$ is then in state $T_i^S \otimes T_i^M$. (The standard theory ignores the possibility that M_q contains subsystems of the same type as S .)
2. A quantum interaction between S and M_q suitably entangles them. Such an interaction can be theoretically represented by unitary map \bar{U} , called *measurement coupling*, but more general forms of evolution are possible (see, e.g., [9]). There is then mathematically well-defined evolution of system $S + M_q$ during a finite time interval. The end result of the evolution is

$$\bar{U}(T_i^S \otimes T_i^M)\bar{U}^\dagger .$$

3. Finally, *reading* the meter gives some value $r \in \mathbb{R}$ of the measured quantity. Observational facts motivate one of the most important assumptions of the standard theory: Each individual reading gives a definite value. This has been called *objectification requirement* [9]. The most important assumption of the standard theory is that, after the reading of value r in an individual measurement, the object system S is in a well-defined state,

$$T_{\text{out},r}^S ,$$

called *conditional* or *selective* state. In quantum mechanics, the fact that an arbitrary large number of identical copies of S is available enables to carry out a whole ensemble of equivalent individual measurements. If the equivalent measurements are repeated many times independently from each other, each reading $r \in \mathbb{R}$ occurs with a definite frequency, P_r . (A more general assumption can be adopted: the final state is conditional on reading the registered value within Borel set X , etc., see [9].)

A special case of a conditional state is given by Dirac's postulate:

A measurement always causes a system to jump in an eigenstate of the observed quantity.

Such a measurement is called *projective* and it is the particular case when $T_{\text{out},r}^S = |r\rangle\langle r|$ where $|r\rangle$ is an eigenvector of a s.a. operator for an eigenvalue r .

The average of all conditional states after registrations, a proper mixture,

$$\sum_r {}_sP_r T_{\text{out},r}^S ,$$

is called *unconditional* or *non-selective* state (the sign “+_s” means that the sum represents a proper mixture of states, see Section 1.3). It is usually described by the words: “make measurements but ignore the results”. One also assumes that

$$\sum_r P_r \mathsf{T}_{\text{out},r}^S = tr^M \left(\bar{\mathsf{U}}(\mathsf{T}_i^S \otimes \mathsf{T}_i^M) \bar{\mathsf{U}}^\dagger \right), \quad (5.1)$$

where tr^M denotes a partial trace defined by any orthonormal frame in the Hilbert space of the meter.

In the standard theory, the reading is a mysterious procedure. If the meter is considered as a quantum system then to observe it, another meter is needed, to observe this, still another is and the resulting series of measurements is called *von-Neumann chain*. At some (unknown) stage including the processes in the mind (brain?) of observer, there is the so-called *Heisenberg cut* that gives the definite value r . Moreover, the conditional state cannot, in general, result by a unitary evolution. The transition

$$\bar{\mathsf{U}}(\mathsf{T}_i^S \otimes \mathsf{T}_i^M) \bar{\mathsf{U}}^\dagger \mapsto \sum_r +_s (P_r \mathsf{T}_{\text{out},r}^S \otimes \mathsf{T}_{\text{out},r}^M), \quad (5.2)$$

where $\mathsf{T}_{\text{out},r}^M$ is the signal state of the meter indicating that outcome of the registration is r , is not a unitary transformation and its existence contradicts the dynamical principles that have been postulated for quantum mechanics. This is the state reduction.

The standard theory is deliberately left incomplete. First, the time and location of the Heisenberg cut is not known. Second, if there are two different kinds of dynamics in quantum mechanics, there ought to be also objective conditions under which each of them is applicable. The standard theory of measurement identifies no such *objective* conditions. It is just assumed to be valid under the subjective condition that a physicist performs a measurement. The difference between the physical process of the measurement and other physical processes remains obscure.

The standard theory ignores such questions and focuses on what effectively happens with the object system. A general measurement is then phenomenologically described by two mathematical quantities. The first is a *state transformer* $\mathcal{O}_r : \mathbf{T}(\mathbf{H}^S) \mapsto \mathbf{L}_r(\mathbf{H}^S)$. \mathcal{O}_r enables us to calculate $\mathsf{T}_{\text{out},r}^S$ from T_i^S by

$$\mathsf{T}_{\text{out},r}^S = \frac{\mathcal{O}_r(\mathsf{T}_i^S)}{tr \left(\mathcal{O}_r(\mathsf{T}_i^S) \right)}.$$

\mathcal{O}_r is a so-called *completely positive map*. Such maps have the form [45]

$$\mathcal{O}_r(\mathsf{T}) = \sum_k \mathsf{O}_{rk} \mathsf{T} \mathsf{O}_{rk}^\dagger \quad (5.3)$$

for any state operator T , where O_{rk} are some operators satisfying

$$\sum_{rk} O_{rk}^\dagger O_{rk} = 1 .$$

Eq. (5.3) is called *Kraus representation*. A given state transformer \mathcal{O}_r does not determine, via Eq. (5.3), the operators O_{rk} uniquely.

The second quantity is an operator E_r called *effect* giving the probability to read value r by

$$P_r = \text{tr} \left(\mathcal{O}_r(T_i^S) \right) = \text{tr}(E_r T_i^S) .$$

The set $\{E_r\}$ of effects E_r for all $r \in \mathbf{R}$ is a POVM (see [11]). As shown there, every POVM satisfies two conditions: positivity,

$$E_r \geq 0 ,$$

for all $r \in \mathbf{R}$, and normalisation,

$$\sum_{r \in \mathbf{R}} E_r = 1 .$$

One can show that \mathcal{O}_r determines the effect E_r by

$$E_r = \sum_k O_{rk}^\dagger O_{rk} .$$

In the standard theory, the state transformer of a given registration contains all information that is necessary for further analysis and for classification of measurements. Such a classification is given in [80], p. 35. Thus, the formalism of the state transformers and POVMs can be considered as the core of the standard theory.

However, the analysis of Chapter 4 has shown that there is no well-defined decomposition of the system $S + M_q$ into subsystems S and M_q after the registered system and the apparatus become entangled by the interaction if the detector is not *clean*, that is if it contains particles of the same type as the object system. Then, there is neither a well-defined conditional state nor any sense in which Eq. (5.1) can be understood. Something like a conditional state could perhaps be found before the interaction between the measured system and the detector: this part of the measuring process could be called “premeasurement” (see [9]). It seems therefore that the standard theory describes the premeasurement onto which some results of the whole measurement process are “grafted” post hoc via Eq. (5.2), or to the whole measurement if the screens and detectors are clean. But this is a condition that cannot be fulfilled. Even if one starts with a clean apparatus, it becomes very quickly polluted during the measurement.

The standard theory is, so to speak, sufficient for all practical purposes (abbreviation FAPP introduced by John Bell) but, as already explained, it is not complete and not strictly correct. The phenomena of reading a meter and of state reduction need a dynamics similar to that of Eq. (5.2) that is different from the usual unitary evolution. Even if one accepted that Schrödinger equation needs correction under some objective conditions, the theory remains incomplete until such objective conditions are formulated and specific corrections are proposed.

Some attempts to solve these difficulties start from the assumption that the transition (5.2) is not observable because the registration of observables that would reveal the difference is either very difficult or that such observables do not exist. One can then deny that the transition (5.2) really takes place and so assume that the objectification is only apparent (no-collapse scenario). There are three most important no-collapse approaches:

1. Quantum decoherence theory [22, 81, 66]. The idea is that system $S + M_q$ composed of a quantum system and an apparatus cannot be isolated from environment E_q . Then the unitary evolution of $S + M_q + E_q$ leads to a non-unitary evolution of $S + M_q$ that can erase all correlations and interferences from $S + M_q$ hindering the objectification [22, 81, 66] (see discussion in Refs. [13, 17, 8, 9]).
2. Superselection sectors approach [38, 61, 76]. Here, classical properties are described by superselection observables of M_q which commute with each other and with all other observables of M_q . Then, the state of M_q after the measurement is equivalent to a suitable proper mixture.
3. Modal interpretation [8]. One assumes that there is a subset of orthogonal-projection observables that, first, can have determinate values in the state of $S + M_q$ before the registration in the sense that the assumption does not violate contextuality (see e.g. [58], Chapter 7) and second, that one can reproduce all important results of ordinary quantum mechanics with the help of these limited set of observables. Thus, one must require that the other observables are not registered.

Other attempts (collapse scenario) do assume that the reduction is a real process and postulate a new dynamics that leads directly to something analogous to (5.2) accepting the consequence that some measurement could disprove this postulate. An example of the collapse scenario is known as Dynamical Reduction Program [18, 57]. It postulates new universal, unique dynamics that is non-linear and stochastic. Both the unitary evolution and the state reduction result as some approximations. The physical idea is that of spontaneous localisation. That is, linear superpositions

of different positions spontaneously decay, either by jumps [18] or by continuous transitions [57]. The form of this decay is chosen judiciously to take a very long time for microsystems, so that the standard quantum mechanics is a good approximation, and a very short time for macrosystems, so that a state reduction results. In this way, a simple explanation of the definite positions of macroscopic systems and of the pointers of registration apparatuses is achieved.

One of the important ideas of the Dynamical Reduction Program is to make the state reduction well-defined by choosing a particular frame for it: the Q -representation. This leads to breaking of the symmetry with respect to all unitary transformations that was not only a beautiful but also a practical feature of standard quantum mechanics.

Another example of collapse scenario is our approach (see Refs. [27, 36, 29, 31]). Its aim is to postulate the existence of state reductions that does not break the unitary symmetry (even if it itself is a non-unitary state transformation) and to formulate hypotheses about the conditions, origin and form of state reduction.

The above review of attempts given is rather short and incomplete. However, the only purpose of it is to specify the position that is taken by the attempt of the present book among them and to focus on our ideas, which are rather different from all others.

5.2 Reformulation of the standard theory

This section explains our approach with the help of two models. It starts by trying to make the theory compatible with the results of Chapters 3 and 4. These results suggest that it is difficult or impossible to identify the state of the measured system after the measurement. Then, the notions of the conditional state and of the state transformer become problematic. We must, therefore, avoid the need for the definite state of the registered system after the registration and replace the definition of the state reduction given by Eq. (5.2) by an equation where the conditional state does not appear.

5.2.1 Stern-Gerlach story retold

Hence, we have to modify the textbook description (e.g., [58], pp. 14 and 375 or [1], p. 230) of the Stern-Gerlach experiment. There are two changes. First, we take more seriously the role of real detectors in the experiment. The detector is assumed to be an object with both classical and quantum model that gives information on the registered quantum object via its classical properties. Hence, it has to satisfy the assumptions of Section 2.1 on classical properties. Second, the description is made

compatible with the consequences of the exchange symmetry for the measurement process that were explained in Chapter 4 so that it can make use of changes of separation status.

The original experiment measures the spin of silver atoms. A silver atom consists of 47 protons and 61 neutrons in the nucleus and of 47 electrons around it. This leads to some complications that can be dealt with technically but that would obscure the ideas we are going to illustrate. Just constructing a simple model, we replace the silver atom by a neutral spin $1/2$ particle.

Let the particle be denoted by S and its Hilbert space by \mathbf{H} . Let \vec{x} be its position, \vec{p} its momentum and S_z the z -component of its spin with eigenvectors $|j\rangle$ and eigenvalues $j\hbar/2$, where $j = \pm 1$ (see e.g. [1], Section 7.4).

Let \mathcal{M} be a Stern-Gerlach apparatus (see Section 1.3) with an inhomogeneous magnetic field oriented so that it separates different z -components of spin of S arriving there. To calculate the evolution of S in the magnetic field, we use the modified Schrödinger equation that describes the interaction between the particle and external field, as it is done, e.g., in [58], p. 375.

Let the detector of the apparatus be a photo-emulsion film \mathcal{D} with energy threshold E_0 . Its emulsion grains are not macroscopic in the sense that each would contain about 10^{23} molecules—they contain only about 10^{10} in average. Still, the chemical and thermodynamic process in them can be described with a sufficient precision by classical chemistry and phenomenological thermodynamics. They have classical states and classical properties. The emulsion grains that are hit by S run through a process of change and of modification and the modification can be made directly visible. \mathcal{D} is a macroscopic object formed by such grains. Let its classical model be D_c and its quantum one be D_q with Hilbert space $\mathbf{H}^{\mathcal{D}}$. According to our theory of classical properties in Chapter 3, the quantum states of the grains, and so of the whole D_q , must be some high-entropy states. The usual description of meters by wave functions is thus not completely adequate.

First, let S be prepared at time t_1 in a definite spin-component state,

$$|\text{in}, j\rangle = |\vec{p}, \Delta\vec{p}\rangle \otimes |j\rangle, \quad (5.4)$$

where $|\vec{p}, \Delta\vec{p}\rangle$ is a Gaussian wave packet with the expectation value \vec{p} and variance $\Delta\vec{p}$ of momentum. To make the mathematics easier, we shall also work with the formalism of wave functions and kernels explained in Section 4.1. The wave function of state (5.4) in an arbitrary representation will be denoted by $\psi_j(\lambda)$. Let system D_q be prepared in metastable state $\mathbf{T}^{\mathcal{D}}$ at t_1 . We assume that D_q consists of N particles of which N_1 are indistinguishable from S . Hence, the kernel of $\mathbf{T}^{\mathcal{D}}$ is

$$T^{\mathcal{D}}(\lambda^{(1)}, \dots, \lambda^{(N_1)}, \lambda^{(N_1+1)}, \dots, \lambda^{(N)}; \lambda^{(1)'}, \dots, \lambda^{(N_1)'}, \lambda^{(N_1+1)'}, \dots, \lambda^{(N)'}) ,$$

where the function $T^{\mathcal{D}}$ is antisymmetric both in variables $\lambda^{(1)}, \dots, \lambda^{(N_1)}$ and $\lambda^{(1)'}, \dots, \lambda^{(N_1)'}$. The initial state of the composite $S + D_q$ then is

$$\bar{T}_j = N_{\text{exch}}^2 \bar{\Pi}_-^{N_1+1} \left(\psi_j(\lambda^{(0)}) \psi_j^*(\lambda^{(0)'}) T^{\mathcal{D}}(\lambda^{(1)}, \dots, \lambda^{(N)}; \lambda^{(1)'}, \dots, \lambda^{(N)'}) \right) \bar{\Pi}_-^{N_1+1}, \quad (5.5)$$

where $\bar{\Pi}_-^{N_1+1}$ denotes the antisymmetrisation in the variables $\lambda^{(0)}, \dots, \lambda^{(N_1)}$ (or $\lambda^{(0)'}, \dots, \lambda^{(N_1)'}$). It is an orthogonal projection acting on Hilbert space $\mathbf{H} \otimes \mathbf{H}^{\mathcal{D}}$ (see Section 4.1).

We also assume that the direction of \vec{p} is suitably restricted and its magnitude respects the energy threshold E_0 . Such states lie in the domain of the apparatus \mathcal{M} , see Section 4.4. According to our theory of meters in Chapter 4, states in the domain of \mathcal{M} have a separation status before their registration by \mathcal{M} . Hence, state (5.4) has a separation status at t_1 and so the system S represents initially an individual quantum object with an objective state. From Definition 4.4 of separation status, it follows that

$$\int d\lambda^{(k)} \psi_j^*(\lambda^{(k)}) T^{\mathcal{D}}(\lambda^{(1)}, \dots, \lambda^{(N_1)}; \lambda^{(1)'}, \dots, \lambda^{(N_1)'}) = 0 \quad (5.6)$$

for any $k = 1, \dots, N_1$, and

$$\int d\lambda^{(l)'} \psi_j(\lambda^{(l)'}) T^{\mathcal{D}}(\lambda^{(1)}, \dots, \lambda^{(N_1)}; \lambda^{(1)'}, \dots, \lambda^{(N_1)'}) = 0 \quad (5.7)$$

for any $l = 1, \dots, N_1$.

Eqs. (5.6) and (5.7) enable us to rewrite state (5.5) in a more explicit form. To this aim, we need the following Lemma:

Lemma 5.1 *Let $F_n(\lambda^{(1)}, \dots, \lambda^{(N)})$, $n = 1, \dots, K$, be K functions of N variables that satisfy:*

1. *Function F_n is antisymmetric in the variables $\lambda^{(1)}, \dots, \lambda^{(N_1)}$ for all n and for some $N_1 < N$.*
2. *For some functions $\psi_j(\lambda)$, $j = 1, \dots, L$, such that $\int d\lambda \psi_j^*(\lambda) \psi_j(\lambda) = 1$,*

$$\int d\lambda^{(k)} \psi_j(\lambda^{(k)}) F_n(\lambda^{(1)}, \dots, \lambda^{(N)}) = 0 \quad (5.8)$$

for all j , n and $k = 1, \dots, N_1$.

3. *$\{F_n\}$ is an orthonormal set,*

$$\int d^N \lambda F_{n'}^*(\lambda^{(1)}, \dots, \lambda^{(N)}) F_n(\lambda^{(1)}, \dots, \lambda^{(N)}) = \delta_{nn'} \quad (5.9)$$

for all n, n' .

Let function \bar{F}_{jn} of $N + 1$ variables $\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(N)}$ be defined by

$$\bar{F}_{jn}(\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(N)}) = \frac{1}{\sqrt{N_1 + 1}} \sum_{k=0}^{N_1} (-1)^{kN_1} \psi_j(\lambda^{(k)}) F_n[\lambda^{(0)} \mapsto \lambda^{(k)}] , \quad (5.10)$$

where

$$F_n[\lambda^{(0)} \mapsto \lambda^{(k)}] = F_n(\lambda^{(k+1)}, \dots, \lambda^{(N_1)}, \lambda^{(0)}, \dots, \lambda^{(k-1)}, \lambda^{(N_1+1)}, \dots, \lambda^{(N)}) .$$

Then functions \bar{F}_{jn} are antisymmetric in variables $\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(N_1)}$ and satisfy:

$$\int d^{N+1} \lambda \bar{F}_{jn}^*(\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(N)}) \bar{F}_{jn'}(\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(N)}) = \delta_{nn'} \quad (5.11)$$

for all j, n and n' .

The set $\lambda^{(a)}, \dots, \lambda^{(b)}$ for any integers a and b is empty if $a > b$ and contains all entries $\lambda^{(c)}$ for $a \leq c \leq b$ in the increasing index order if $a \leq b$.

Proof Function \bar{F}_{jn} is antisymmetric because F_n is and the sum in (5.10) contains already exchanges of $\lambda^{(0)}$ and $\lambda^{(k)}$ for all $k > 0$ with the proper signs (see Section 4.1). To show Eq. (5.11), we substitute Eq. (5.10) into the right-hand side of Eq. (5.11):

$$\begin{aligned} \int d^{N+1} \lambda \bar{F}_{jn'}^* \bar{F}_{jn} &= \frac{1}{N_1 + 1} \int d^{N+1} \lambda \sum_{k=0}^{N_1} (-1)^{kN_1} \sum_{l=0}^{N_1} (-1)^{lN_1} \\ &\quad \times \psi_j(\lambda^{(k)}) F_n[\lambda^{(0)} \mapsto \lambda^{(k)}] \psi_j^*(\lambda^{(l)}) F_{n'}^*[\lambda^{(0)} \mapsto \lambda^{(l)}] . \end{aligned}$$

The terms

$$\int d^{N+1} \lambda \psi_j(\lambda^{(k)}) F_n[\lambda^{(0)} \mapsto \lambda^{(k)}] \psi_j^*(\lambda^{(l)}) F_{n'}^*[\lambda^{(0)} \mapsto \lambda^{(l)}]$$

vanish for any $k \neq l$ because of Eq. (5.8). The remaining terms

$$\int d^{N+1} \lambda \psi_j(\lambda^{(k)}) F_n[\lambda^{(0)} \mapsto \lambda^{(k)}] \psi_j^*(\lambda^{(k)}) F_{n'}^*[\lambda^{(0)} \mapsto \lambda^{(k)}]$$

are equal to $\delta_{nn'}$ for all k because of the normalisation of ψ_j and Eq. (5.9), **QED**.

State (5.5) has then the following kernel:

$$\begin{aligned} \bar{T}_j(\lambda^{(0)}, \dots, \lambda^{(N)}; \lambda^{(0)'}, \dots, \lambda^{(N)'}) &= \frac{1}{N_1 + 1} \sum_{k=0}^{N_1} (-1)^{kN_1} \sum_{l=0}^{N_1} (-1)^{lN_1} \psi_j(\lambda^{(k)}) \psi_j^*(\lambda^{(l)'}) \\ &\quad T^{\mathcal{D}}(\lambda^{(k+1)}, \dots, \lambda^{(N_1)}, \lambda^{(0)}, \dots, \lambda^{(k-1)}, \lambda^{(N_1+1)}, \dots, \lambda^{(N)}; \\ &\quad \lambda^{(l+1)'}, \dots, \lambda^{(N_1)'}, \lambda^{(0)'}, \dots, \lambda^{(l-1)'}, \lambda^{(N_1+1)'}, \dots, \lambda^{(N)'}) . \end{aligned} \quad (5.12)$$

Kernel \bar{T}_j can be shown to be antisymmetric in variables $\lambda^{(0)}, \dots, \lambda^{(N_1)}$ and $\lambda^{(0)'}, \dots, \lambda^{(N_1)'}$ and to have trace equal 1 by the same methods as those used to prove Lemma 5.1. Eqs. (5.6) and (5.7) expressing the separation status of $|\psi\rangle$ play an important role in the derivation of formula (5.12).

The initial state of $S + D_q$ does not contain any modified emulsion grains. Extremal states with this property form a subspace of the Hilbert space $\bar{\Pi}^{N_1+1}(\mathbf{H} \otimes \mathbf{H}^{\mathcal{D}})$ of $S + D_q$. Let us denote the projection to this subspace by $\bar{\Pi}[\emptyset]$. Thus, we have

$$\text{tr}(\bar{T}_j \bar{\Pi}[\emptyset]) = 1. \quad (5.13)$$

The process of registration includes the interaction of S with the magnetic field and with system D_q as well as the resulting modification of the emulsion grains. We assume that meter \mathcal{M} is *ideal*: each copy of S that arrives at the emulsion D_q modifies at least one emulsion grain.

The registration is assumed to be a quantum evolution described by a unitary group $\bar{U}(t)$, the so-called *measurement coupling* see [9]. We assume that $\bar{U}(t)$ commutes with $\bar{\Pi}_-^{N_1+1}$, see Section 4.1. Let t_2 be the time at which the modification of the hit grains is finished and let $\bar{U} = \bar{U}(t_2 - t_1)$. We are going to derive some important properties of $\bar{U} \bar{T}_j \bar{U}^\dagger$, and for this we need a technical trick that transforms calculations with kernels into that with wave functions.

Let

$$\mathbb{T}^{\mathcal{D}} = \sum_n a_n |n\rangle \langle n| \quad (5.14)$$

be the spectral decomposition of $\mathbb{T}^{\mathcal{D}}$. Then, $0 \leq a_n \leq 1$ for each $n \in \mathbb{N}$ and $\sum_n a_n = 1$. In λ -representation, state $|n\rangle$ has the wave function $\varphi_n(\lambda^{(1)}, \dots, \lambda^{(N)})$.

Eqs. (5.12) and (5.14) imply that

$$\bar{T}_j(\lambda^{(0)}, \dots, \lambda^{(N)}; \lambda^{(0)'}, \dots, \lambda^{(N)'}) = \sum_n a_n \bar{\Psi}_{jn}(\lambda^{(0)}, \dots, \lambda^{(N)}) \bar{\Psi}_{jn}^*(\lambda^{(0)'}, \dots, \lambda^{(N)'}) , \quad (5.15)$$

where

$$\bar{\Psi}_{jn}(\lambda^{(0)}, \dots, \lambda^{(N)}) = \frac{1}{\sqrt{N_1 + 1}} \sum_{k=0}^{N_1} (-1)^{kN_1} \psi_j(\lambda^{(k)}) \varphi_n[\lambda^{(0)} \mapsto \lambda^{(k)}] . \quad (5.16)$$

Lemma 5.2 *Eq. (5.15) is the spectral decomposition of state \bar{T}_j .*

Proof Conditions (5.6) and (5.7) on $|\psi_j\rangle$ and $\bar{\mathbb{T}}^{\mathcal{D}}$ imply

$$\sum_n a_n \int d\lambda^{(k)} \int d\lambda^{(k)'} \psi_j^*(\lambda^{(k)}) \psi_j(\lambda^{(k)'}) \varphi_n(\lambda^{(1)}, \dots, \lambda^{(N)}) \varphi_n^*(\lambda^{(1)'}, \dots, \lambda^{(N)'}) = 0$$

for all $k = 1, \dots, N_1$. However, the integral defines a positive kernel

$$K_n(\lambda^{(1)}, \dots, \lambda^{(k-1)} \lambda^{(k+1)}, \dots, \lambda^{(N)}; \lambda^{(1)'}, \dots, \lambda^{(k-1)'} \lambda^{(k+1)'}, \dots, \lambda^{(N)'})$$

for each n and a sum with positive coefficients of such kernels can be zero only if each such kernel itself vanishes. Hence, we have

$$\int d\lambda_k \psi^*(\lambda^{(k)}) \varphi_n(\lambda^{(1)}, \dots, \lambda^{(N)}) = 0 \quad (5.17)$$

for each n and all $k = 1, \dots, N$.

From Lemma 5.1, it then follows now that

$$\langle \bar{\Psi}_{jn} | \bar{\Psi}_{jn'} \rangle = \delta_{nn'} .$$

This implies Lemma 5.2, **QED**.

A simple consequence of Lemma 5.2 is the following. Combining Eqs. (5.13) and (5.15), we obtain

$$tr \left(\sum_n a_n |\bar{\Psi}_{jn}\rangle \langle \bar{\Psi}_{jn}| \bar{\Pi}[\emptyset] \right) = \sum_n a_n tr \left((\bar{\Pi}[\emptyset] |\bar{\Psi}_{jn}\rangle) (\langle \bar{\Psi}_{jn}| \bar{\Pi}[\emptyset]) \right) = 1 .$$

But operator $\bar{\Pi}[\emptyset] |\bar{\Psi}_{jn}\rangle \langle \bar{\Psi}_{jn}| \bar{\Pi}[\emptyset]$ is positive so that its trace must be non-negative. As the sum of a_n 's is already 1, we must have

$$tr(\bar{\Pi}[\emptyset] |\bar{\Psi}_{jn}\rangle \langle \bar{\Psi}_{jn}| \bar{\Pi}[\emptyset]) = 1$$

or

$$\langle \bar{\Pi}[\emptyset] | \bar{\Psi}_{jn} | \bar{\Psi}_{jn} | \bar{\Pi}[\emptyset] \rangle = 1$$

for each n . However,

$$|\bar{\Psi}_{jn}\rangle = \bar{\Pi}[\emptyset] |\bar{\Psi}_{jn}\rangle + (1 - \bar{\Pi}[\emptyset]) |\bar{\Psi}_{jn}\rangle$$

and

$$\langle \bar{\Pi}[\emptyset] | \bar{\Psi}_{jn} | (1 - \bar{\Pi}[\emptyset]) | \bar{\Psi}_{jn} \rangle = 0$$

so that

$$1 = \langle \bar{\Psi}_{jn} | \bar{\Psi}_{jn} \rangle = \langle \bar{\Pi}[\emptyset] | \bar{\Psi}_{jn} | \bar{\Pi}[\emptyset] | \bar{\Psi}_{jn} \rangle + \langle (1 - \bar{\Pi}[\emptyset]) | \bar{\Psi}_{jn} | (1 - \bar{\Pi}[\emptyset]) | \bar{\Psi}_{jn} \rangle .$$

Hence,

$$\bar{\Pi}[\emptyset] |\bar{\Psi}_{jn}\rangle = |\bar{\Psi}_{jn}\rangle . \quad (5.18)$$

Let us now return to the time evolution of \bar{T}_j within $\bar{\Pi}_-^{N_1+1}(\mathbf{H} \otimes \mathbf{H}^{\mathcal{D}})$ from t_1 to t_2 . System $S + D_q$ is composed of two disjoint subsystems, S' and D'_q , S' containing S

and all N_1 particles of D_q that are indistinguishable from S . Then, $\bar{\Pi}_-^{N_1+1}(\mathbf{H} \otimes \mathbf{H}^{\mathcal{D}}) = (\mathbf{H})_-^{N_1+1} \otimes \mathbf{H}^{\mathcal{D}'}$. The evolution defines states $\bar{\mathsf{T}}_j(t_2)$ of $S + D_q$ by:

$$\bar{U} \bar{\mathsf{T}}_j \bar{U}^\dagger = \bar{\mathsf{T}}_j(t_2) . \quad (5.19)$$

Evolution \bar{U} includes a thermodynamic relaxation of $S + D_q$ and a loss of separation status of S . Thus, in general, quantum system S does not represent an individual quantum object after the registration. The individual states that could be ascribed to S as its objective properties are not well defined (see Section 4.1) at $t = t_2$. We can say that they do not exist. However, the whole composite $S + D_q$ is a quantum object, prepared in the measurement experiment, hence one can consider its individual states as its objective properties.

Accordingly, states $\bar{\mathsf{T}}_j(t_2)$ also describe the modified emulsion grains, which can be called *detector signals*. The signals are concentrated within two strips of the film, each strip corresponding to one value of j . The two space regions, R_+ and R_- , of the two strips are sufficiently separated and help to determine, in the present case, what is generally called a pointer observable: the occurrence of a modified emulsion grain within R_+ or R_- . Let the projections onto the subspaces of $(\mathbf{H})_-^{N_1+1} \otimes \mathbf{H}^{\mathcal{D}'}$ containing the corresponding extremal states be $\bar{\Pi}[R_j]$.

We avoid specifying $\bar{U}(t)$ e.g. by writing the Hamiltonian of system $S + \mathcal{D}_q$. Instead, we express the condition that the meter registers \mathbf{S}_z through properties of end states $\mathsf{T}_j(t_2)$ as follows:

$$\text{tr} \left(\bar{\Pi}[R_j] \bar{\mathsf{T}}_k(t_2) \right) = \delta_{jk} . \quad (5.20)$$

If we substitute Eqs. (5.19) and (5.15) into (5.20), we obtain

$$\sum_n a_n \text{tr}(\bar{U} |\bar{\Psi}_{kn}\rangle \langle \bar{\Psi}_{kn}| \bar{U}^\dagger \bar{\Pi}[R_j]) = \delta_{jk} .$$

By the same argument as that leading to formula (5.18), we then have

$$\bar{\Pi}[R_j] |\bar{\Psi}_{kn}(t_2)\rangle = \delta_{jk} |\bar{\Psi}_{kn}(t_2)\rangle , \quad (5.21)$$

where

$$|\bar{\Psi}_{kn}(t_2)\rangle = \bar{U} |\bar{\Psi}_{kn}\rangle .$$

Hence, the state $\bar{U} |\bar{\Psi}_{kn}\rangle$ contains modified emulsion grains in the region R_k and no such grains in the region R_l for each n and $l \neq k$.

Suppose next that the initial state of S at t_1 is

$$|\text{in}\rangle = \sum_j c_j |\text{in}, j\rangle \quad (5.22)$$

with

$$\sum_j |c_j|^2 = 1 .$$

The linearity of $\bar{\mathbf{U}}$ implies the following form of the corresponding end state $\bar{\mathbf{T}}(t_2) \in \mathbf{T}(\bar{\Pi}_-^{N_1+1}(\mathbf{H} \otimes \mathbf{H}^{\mathcal{D}}))$:

$$\begin{aligned} \bar{\mathbf{T}}(t_2) &= N_{\text{exch}}^2 \bar{\mathbf{U}} \bar{\Pi}_-^{N_1+1} \left[\left(\sum_j c_j |\text{in}, j\rangle \right) \left(\sum_{j'} c_{j'}^* \langle \text{in}, j'| \right) \otimes \mathbf{T}^{\mathcal{D}} \right] \bar{\Pi}_-^{N_1+1} \bar{\mathbf{U}}^\dagger \\ &= \sum_{jj'} c_j c_{j'}^* \bar{\mathbf{T}}_{jj'}(t_2) , \end{aligned} \quad (5.23)$$

Operators $\bar{\mathbf{T}}_{jj'}(t_2)$ act on the Hilbert space $\bar{\Pi}_-^{N_1+1}(\mathbf{H} \otimes \mathbf{H}^{\mathcal{D}})$ of $S + D_q$ and are defined by

$$\bar{\mathbf{T}}_{jj'}(t_2) = N_{\text{exch}} \bar{\mathbf{U}} \bar{\Pi}_-^{N_1+1} (|\text{in}, j\rangle \langle \text{in}, j'| \otimes \mathbf{T}^{\mathcal{D}}) \bar{\Pi}_-^{N_1+1} \bar{\mathbf{U}}^\dagger . \quad (5.24)$$

They are state operators only for $j' = j$. Eqs. (5.19) and (5.5) imply that

$$\mathbf{T}_{jj}(t_2) = \mathbf{T}_j(t_2) .$$

If we substitute the spectral decomposition (5.14) of $\mathbf{T}^{\mathcal{D}}$ into Eq. (5.24), we obtain for the kernel of operator $\mathbf{T}_{jj'}(t_2)$

$$\begin{aligned} T_{jj'}(t_2) &= \sum_n a_n \bar{\mathbf{U}} \\ &\times \left(\sum_{k=0}^{N_1} (-1)^{N_1 k} \psi_j(\lambda^{(k)}) \varphi_n(\lambda^{(k+1)}, \dots, \lambda^{(N_1)}, \lambda^{(0)}, \dots, \lambda^{(k-1)}, \lambda^{(N_1+1)}, \dots, \lambda^{(N)}) \right) \\ &\times \left(\sum_{l=0}^{N_1} (-1)^{N_1 l} \psi_{j'}^*(\lambda^{(l)}) \varphi_n^*(\lambda^{(l+1)'}, \dots, \lambda^{(N_1)'}, \lambda^{(0)'}, \dots, \lambda^{(l-1)'}, \lambda^{(N_1+1)'}, \dots, \lambda^{(N)'}) \right) \bar{\mathbf{U}}^\dagger \\ &= \sum_n a_n \left(\bar{\mathbf{U}} \bar{\Psi}_{jn}(\lambda^{(0)}, \dots, \lambda^{(N)}) \right) \left(\bar{\Psi}_{j'n}^*(\lambda^{(0)'}, \dots, \lambda^{(N)'}) \bar{\mathbf{U}}^\dagger \right) , \end{aligned}$$

or

$$\mathbf{T}_{jj'}(t_2) = \sum_n a_n |\bar{\Psi}_{jn}(t_2)\rangle \langle \bar{\Psi}_{j'n}(t_2)| . \quad (5.25)$$

Eq. (5.25) is, of course, not the spectral decomposition of $\mathbf{T}_{jj'}(t_2)$ because this operator is not self-adjoint, but it can be used to show that Eq. (5.21) implies:

$$\text{tr} \left(\bar{\Pi}[R_k] |\bar{\Psi}_{jn}(t_2)\rangle \langle \bar{\Psi}_{j'n}(t_2)| \right) = \delta_{kj} \delta_{kj'} . \quad (5.26)$$

Then, because of the orthonormality of state vectors $|\bar{\Psi}_{jn}(t_2)\rangle$, it follows that

$$\text{tr} \left(\bar{\Pi}[R_j] \bar{\mathbf{T}}_{kl}(t_2) \right) = \delta_{jk} \delta_{jl} \quad (5.27)$$

and

$$\text{tr}\left(\bar{\Pi}[R_j]\bar{T}(t_2)\right) = |c_j|^2 . \quad (5.28)$$

The significance of Eq. (5.28) is that the modified grains will be found in the strip j with the probability given by the Born rule for registering the spin j in the state (5.22).

Eq. (5.23) can be written as

$$\bar{T}(t_2) = \bar{T}_{\text{end1}} + \bar{T}_{\text{end0}} , \quad (5.29)$$

where

$$\bar{T}_{\text{end1}} = \sum_j |c_j|^2 \bar{T}_j(t_2) , \quad \bar{T}_{\text{end0}} = \sum_{j \neq j'} c_j c_{j'}^* \bar{T}_{jj'}(t_2) . \quad (5.30)$$

It follows that

$$\text{tr}(\bar{T}_{\text{end1}}) = 1 , \quad \text{tr}(\bar{T}_{\text{end0}}) = 0 . \quad (5.31)$$

Eq. (5.30) says that \bar{T}_{end1} is a convex combination of quantum states that differ from each other by expectation values of operator $\bar{\Pi}[R_j]$.

Finally, we have to analyse more closely what is observed in Stern-Gerlach experiment. The basic fact is that there are modified emulsion grains at some definite positions at the film after each registration. This is represented by definite states of the classical model D_c of the film. A basic assumption about classical models is that their states are objective, that is, they exist before being observed and the observation only reveals them (see Chapter 2). A state T_c of D_c can be described by specifying the positions of the modified grains. Then we can express the fact that the modified grains lie in strip R_j by the (epistemic) classical state represented by composed expression $T_c \subset R_j$. Quantum mechanics can only give us the probabilities $P(T_c)$ that state T_c is observed:

$$P(T_c) = \text{tr}\left(\bar{T}(t_2)\bar{\Pi}[R_j]\right) .$$

According to Minimum Interpretation, state $\bar{T}(t_2)$ just describes the statistics of the ensemble of particular measurements on system $S + D_q$ and does not refer to anything existing before the registration and concerning each individual system. Minimum Interpretation does not even say that a prepared individual system is an element of a definite ensemble of measurements: such ensemble is only defined if the registered observable is fixed.

According to RCU Interpretations, state $\bar{T}(t_2)$ is a property referring directly to each individual composite system $S + D_q$ immediately before the registration. Moreover, two quantum states $\bar{T}_j(t_2)$, $j = 1, 2$, are in a bijective relation with two classical states $T_c \subset R_j$, $j = 1, 2$. The observation that the classical state of D_c is $T_c \subset R_j$ implies, therefore, that the quantum state of $S + D_q$ must be $\bar{T}_j(t_2)$ already

before the (classical) observation. Hence, the state of the individual composite system $S + D_q$ immediately before the registration must be a proper mixture of states $\bar{T}_j(t_2)$ each of which has a definite value of j :

$$\sum_j +_s |c_j|^2 \bar{T}_j(t_2) \quad (5.32)$$

instead of (5.23) that results by unitary, linear evolution law of quantum mechanics. Observe that the transition from state (5.23) to (5.32) is non-linear but preserving the norm of the state. The additional “evolution” from state (5.23) to state (5.32) that must then be caused in some way by the registration, is the state reduction.

This section was rather technical because it was to describe registrations in a way that avoids the concept of conditional state. This lead to a new formalism represented by Eqs. (5.29) and (5.32). The state reduction is now defined as the transition between the states described by these two equations.

5.2.2 Screen

Screens are used in most preparation procedures. For example, in optical experiments [64], polarisers, such as Glan-Thompson ones, are employed. A polariser contains a crystal that decomposes the coming light into two orthogonal-polarisation parts. One part disappears inside an absorber and the other is left through. Similarly, the Stern-Gerlach experiment (see Section 2.3) can be modified so that the beam corresponding to spin down is blocked out by an absorber and the other beam is left through. In the interference experiment [73], there are several screens, which are just walls with openings. Generally, a screen is a macroscopic body that decomposes the incoming, already prepared, beam into one part that disappears inside the body and the other that goes through.

Here, a simple model of screen is constructed and its physics is studied. Let the particle S interacting with the screen have mass μ and spin 0 and the screen have the following geometry:

Assumption 5.1 *The screen is at $x_3 = 0$ and the half-spaces $x_3 < 0$ and $x_3 > 0$ are empty. There is a opening D in the screen, that is D is an open subset of the plane $x_3 = 0$, not necessary connected (e.g., two slits). Finally, let the screen be stationary, that is the geometry is time independent.*

For the interaction between the particle and the screen, we assume:

Assumption 5.2 *Inside the half-spaces $x_3 < 0$, $x_3 > 0$, the wave function $\psi(\vec{x}, t)$ of S satisfies the free Schrödinger equation,*

$$i\hbar \frac{\partial \psi(\vec{x}, t)}{\partial t} = -\frac{\hbar^2}{2\mu} \left(\frac{\partial^2 \psi(\vec{x}, t)}{\partial x_1^2} + \frac{\partial^2 \psi(\vec{x}, t)}{\partial x_2^2} + \frac{\partial^2 \psi(\vec{x}, t)}{\partial x_3^2} \right) . \quad (5.33)$$

Let us denote the part of the solution $\psi(\vec{x}, t)$ in the left half-space $x_3 < 0$ by $\psi_i(\vec{x}, t)$ and in the right half-space by $\psi_{traf}(\vec{x}, t)$. Let $\psi_i(\vec{x}, t)$ be the $x_3 < 0$ -part of a wave packet with $p_3 > 0$,

$$\psi(\vec{x}, t) = \left(\frac{1}{2\pi\hbar} \right)^{3/2} \int_{\mathbb{R}^3} d^3p \tilde{\psi}(\vec{p}) \exp \left[\frac{i}{\hbar} \left(-\frac{|\vec{p}|^2}{2\mu} t + \vec{p} \cdot \vec{x} \right) \right] , \quad (5.34)$$

where $\tilde{\psi}(\vec{p})$ is a rapidly decreasing function (see [63], p. 133) with $\tilde{\psi}(\vec{p}) = 0$ for all $p_3 \leq 0$, and let, for any fixed (finite) time, function $\psi_{traf}(\vec{x}, t)$ is rapidly decreasing.

At the points of the screen, the wave function is discontinuous. From the left, the boundary values

$$\lim_{x_3 \rightarrow -0} \psi(\vec{x}, t) = \lim_{x_3 \rightarrow -0} \psi_i(\vec{x}, t) , \quad \lim_{x_3 \rightarrow -0} \frac{\partial \psi}{\partial x_3}(\vec{x}, t) = \lim_{x_3 \rightarrow -0} \frac{\partial \psi_i}{\partial x_3}(\vec{x}, t) ,$$

are determined by the solution $\psi_i(\vec{x}, t)$. From the right,

$$\lim_{x_3 \rightarrow 0} \psi_{traf}(\vec{x}, t) = 0 , \quad \lim_{x_3 \rightarrow 0} \frac{\partial \psi_{traf}}{\partial x_3}(\vec{x}, t) = 0 \quad (5.35)$$

for $(x_1, x_2) \notin D$ and

$$\lim_{x_3 \rightarrow 0} \psi_{traf}(\vec{x}, t) = \lim_{x_3 \rightarrow -0} \psi(\vec{x}, t) , \quad \lim_{x_3 \rightarrow 0} \frac{\partial \psi_{traf}}{\partial x_3}(\vec{x}, t) = \lim_{x_3 \rightarrow -0} \frac{\partial \psi}{\partial x_3}(\vec{x}, t) \quad (5.36)$$

for $(x_1, x_2) \in D$.

This expresses the notion that all particles arrive at the screen from the left and those that hit the screen are absorbed by the screen and cannot reappear.

The mathematical problem defined by Assumptions 5.1 and 5.2 can be solved by the same method as the diffraction problem in optics can (see [6], Section 8.3.1)² even if the wave equation is a rather different kind of differential equation than the Schrödinger equation. Indeed, for a monochromatic wave,

$$\psi(\vec{x}, t) = \exp \left(-\frac{i}{\hbar} Et \right) \Psi(\vec{x}) ,$$

Eq. (5.33) implies

$$\Delta \Psi(\vec{x}) + k^2 \Psi(\vec{x}) = 0 ,$$

where

$$k^2 = \frac{2\mu E}{\hbar^2} ,$$

which coincides with Helmholtz equation ([6], p. 375). The solution of Helmholtz equation in the half-space $x_3 > 0$ given by Fresnel-Kirchhof diffraction formula ([6],

²The author is indebted to Pavel Kurasov for clarifying this point.

p. 380) then leads to the general solution $\psi_{\text{traf}}(\vec{x}, t)$ (which is a Fourier integral of monochromatic waves defined by $\tilde{\psi}(\vec{p})$ of Eq. (5.34)) that satisfies the required boundary conditions. Hence, the solution exists and is unique.

We can define absorption, P_{abs} , and transmission, P_{tra} , probabilities for the screen as follows:

$$P_{\text{tra}} = \lim_{t \rightarrow \infty} \int_{\mathbb{R}^2} d^2x \int_0^\infty dx_3 |\psi_{\text{traf}}(\vec{x}, t)|^2 \quad (5.37)$$

and

$$P_{\text{abs}} = 1 - P_{\text{tra}} .$$

This is based on the idea that the initial rapidly decreasing wave packet will leave the left half-space completely for $t \rightarrow \infty$.

Function $\psi_{\text{traf}}(\vec{x}, t)$ is not normalised and its norm is $P_{\text{tra}}^2 < 1$. Hence, the model defines a dynamics that is not unitary. This is clearly due to the incompleteness of the model: particles that hit the screen are absorbed and this part of the process was ignored above. Let us give a short account of the physics of absorption. The body \mathcal{B} is assumed to be a perfect absorber so that S does not leave it. Thus, the screen is assumed to be *ideal*: every particle that arrives at it is either absorbed or goes through the opening. Let the quantum model B_q of \mathcal{B} be a macroscopic quantum system with Hilbert space \mathbf{H}^B (a real screen is somewhat thicker than a plane, but we just construct a model). The process of disappearance of a quantum system S in a macroscopic body B_q can be decomposed into three steps. First, S is prepared in a state that has a separation status so that a further preparation or registration (in which the screen participates) can be made. Second, such S enters B_q and ditch most of its kinetic energy somewhere inside B_q . Third, the energy passed to B_q is dissipated and distributed homogeneously through B_q in a process aiming at thermodynamic equilibrium. Then, system $S + B_q$ cannot be decomposed into well-defined subsystems S and B_q any more, S ceases to be an object and it does not possess any individual state of its own after being absorbed if there are any particle of the same type within B_q , as it has been explained in Section 4.1. S loses its separation status. Even if, originally, no particle of the same type as S is within B_q , in the course of the experiment, B_q will be polluted by many of them.

It is important that the absorption process is (or can be in principle) observable. For instance, the increase of the temperature of B_q due to the energy of the absorbed particles can be measured. That is, either a single particle S has enough kinetic energy to cause an observable temperature change, or there is a cumulative effect of more absorbed particles. More precisely, suppose that the energy E^S of the absorbed particle is small,

$$E^S < \Delta E^B , \quad (5.38)$$

where ΔE^B is the variance of the screen energy in the initial state of the screen so

that it would seem that the absorption could not change the classical state of the screen. However, after a sufficient number of absorptions, the total change of the energy will surpass the limit (5.38) so that the average change of the screen energy due to one absorption is well defined. In any case, the initial and final states of B_q cannot be described by wave functions and they differ by their classical properties from each other, e.g. by the temperature.

Let us now try to complete the model including the process of absorption by writing the initial state as a linear combination of the absorbed and the transmitted ones. We define a function $\psi_{\text{trai}}(\vec{x}, t)$ for $x_3 < 0$ as the solution of Schrödinger equation (5.33) satisfying the boundary conditions

$$\lim_{x_3 \rightarrow 0} \psi_{\text{trai}}(\vec{x}, t) = 0, \quad \lim_{x_3 \rightarrow 0} \frac{\partial \psi_{\text{trai}}}{\partial x_3}(\vec{x}, t) = 0 \quad (5.39)$$

for $(x_1, x_2) \notin D$ and

$$\lim_{x_3 \rightarrow 0} \psi_{\text{trai}}(\vec{x}, t) = \lim_{x_3 \rightarrow 0} \psi_i(\vec{x}, t), \quad \lim_{x_3 \rightarrow 0} \frac{\partial \psi_{\text{trai}}}{\partial x_3}(\vec{x}, t) = \lim_{x_3 \rightarrow 0} \frac{\partial \psi_i}{\partial x_3}(\vec{x}, t) \quad (5.40)$$

for $(x_1, x_2) \in D$.

Then, the pair of functions $\psi_{\text{trai}}(\vec{x}, t)$ and $\psi_{\text{traf}}(\vec{x}, t)$ define a C^1 solution to the Schrödinger equation in the whole space as if the screen did not exist. Let us denote this function by $\sqrt{P_{\text{tra}}} \psi_{\text{tra}}(\vec{x}, t)$. Then, $\psi_{\text{tra}}(\vec{x}, t)$ is a normalised solution running from the left to the right and vanishing in the left-hand half-space for large times.

Finally, let us define function $\psi_{\text{abs}}(\vec{x}, t)$ in the left-hand half-space by

$$\psi(\vec{x}) = c_{\text{tra}} \psi_{\text{tra}}(\vec{x}) + c_{\text{abs}} \psi_{\text{abs}}(\vec{x}), \quad (5.41)$$

where $c_{\text{tra}} = \sqrt{P_{\text{tra}}}$, $c_{\text{abs}} = \sqrt{1 - P_{\text{tra}}}$ and $\psi_{\text{tra}}(\vec{x}, t)$ is a normalised wave function of the part that will be left through and $\psi_{\text{abs}}(\vec{x})$ that that will be absorbed by B_q . Indeed, the two wave functions ψ_{tra} and ψ_{abs} must be orthogonal to each other because their large-time evolution gives $\psi_{\text{abs}} = 0$ in the right-hand half space and $\psi_{\text{tra}} = 0$ in the left-hand half space.

Decomposition (5.41) is determined by the nature of B_q : for a polariser, these are the two orthogonal polarisation states, and for a simple screen consisting of a wall with an opening, these can be calculated from the geometry of B_q .

The initial state of B_q is a high-entropy one (see Chapter 3). It is, therefore, described by a state operator \mathbb{T}_i . Then the initial state for the evolution of the composite is

$$\bar{\mathbb{T}}_i = N_{\text{exch}}^2 \bar{\Pi}_S(|\psi_i\rangle\langle\psi_i| \otimes \mathbb{T}_i) \bar{\Pi}_S,$$

where $N_{\text{exch}}^2 = \text{tr}(\bar{\Pi}_S(|\psi\rangle\langle\psi| \otimes \mathbb{T}_i) \bar{\Pi}_S)$ and $\bar{\Pi}_S$ is the symmetrization or anti-symmetrization over all particles indistinguishable from S (see Section 4.1) within the

composite system $S + B_q$ (we leave open the question of whether they are fermions or bosons—thus we make a more general theory than that of the previous section). It is an operator on the Hilbert space $\mathbf{H} \otimes \mathbf{H}^B$. Further steps are analogous to those for the absorption of the registered system in the photo-emulsion \mathcal{D} that has been analysed in more details in the previous section and we can skip the details here.

Let the evolution of the composite $S + B_q$ be described by operator \bar{U} . It contains the absorption and dissipation process in B_q . \bar{U} is a unitary operator on the Hilbert space $\mathbf{H} \otimes \mathbf{H}^B$ that commutes with projection $\bar{\Pi}_S$ (see Section 4.5) so that it leaves subspace $\bar{\Pi}_S(\mathbf{H} \otimes \mathbf{H}^B)$ invariant and so defines a unitary operator on Hilbert space $\bar{\Pi}_S(\mathbf{H} \otimes \mathbf{H}^B)$ of the composite. It is independent of the choice of the initial state. After the process is finished, we obtain

$$\bar{T}_f = N_{\text{exch}}^2 \bar{\Pi}_S \bar{U}(|\psi_i\rangle\langle\psi_i| \otimes T_i) \bar{U}^\dagger \bar{\Pi}_S .$$

Using decomposition (5.41), we can write

$$\begin{aligned} \bar{T}_f = & N_{\text{exch}}^2 c_{\text{abs}} c_{\text{abs}}^* \bar{\Pi}_S \bar{U}(|\psi_{\text{absi}}\rangle\langle\psi_{\text{absi}}| \otimes T_i) \bar{U}^\dagger \bar{\Pi}_S \\ & + N_{\text{exch}}^2 c_{\text{tra}} c_{\text{tra}}^* \bar{\Pi}_S \bar{U}(|\psi_{\text{traf}}\rangle\langle\psi_{\text{traf}}| \otimes T_i) \bar{U}^\dagger \bar{\Pi}_S \\ & + N_{\text{exch}}^2 c_{\text{tra}} c_{\text{abs}}^* \bar{\Pi}_S \bar{U}(|\psi_{\text{traf}}\rangle\langle\psi_{\text{absi}}| \otimes T_i) \bar{U}^\dagger \bar{\Pi}_S \\ & + N_{\text{exch}}^2 c_{\text{abs}} c_{\text{tra}}^* \bar{\Pi}_S \bar{U}(|\psi_{\text{absi}}\rangle\langle\psi_{\text{traf}}| \otimes T_i) \bar{U}^\dagger \bar{\Pi}_S . \end{aligned} \quad (5.42)$$

The first term describes the process that starts with state ψ_{absi} . Thus, S does not reappear at the end and the result is an excited state \bar{T}'_f of the screen that has absorbed S . The second term represents the evolution that starts with S in the state ψ_{traf} . Then the screen remains in its initial state T_i and S reappears in state ψ_{traf} . Hence,

$$\bar{T}_f = \bar{T}_{\text{end1}} + \bar{T}_{\text{end0}} ,$$

where

$$\bar{T}_{\text{end1}} = |c_{\text{abs}}|^2 \bar{T}'_f + |c_{\text{tra}}|^2 |\psi_{\text{traf}}\rangle\langle\psi_{\text{traf}}| \otimes T_i .$$

State \bar{T}_{end1} is a convex combination of two states that differ from each other by their classical properties while

$$\text{tr}(\bar{T}_{\text{end0}}) = 0 .$$

We can now argue in analogy with the previous section: RCU interpretation suggests together with the observation that only the first two terms describe the true end state of the composite after each individual individual process and the state is not just a convex combination but a proper mixture (see Section 1.3):

$$\bar{T}_{\text{truef}} = P_{\text{tra}} |\psi_{\text{traf}}\rangle\langle\psi_{\text{traf}}| \otimes T_i + P_{\text{abs}} \bar{T}'_f . \quad (5.43)$$

The transition from \bar{T}_f to $\bar{T}_{\text{true}f}$ such a mixture is our definition of state reduction as in Section 5.2.

Again, the state reduction is not a unitary transformation: First, the non-diagonal terms in (5.42) have been erased. Second, we have also assumed that state ψ_{traf} is the state of S that has been *prepared* by the screening. This means for us that it is a real state with a separation status. Hence, operator $\bar{\Pi}_S$ can be left out in Formula (5.43). This is, of course, another violation of unitarity.

The disappearance of S in B_q , as well as the disappearance of S in the photo-emulsion \mathcal{D} described in the previous section, is a physical process that have a definite time and place. This suggests that the state reduction occurs at the time and the place of the possible absorption of the particle in B_q or \mathcal{D} . The possible absorption had to be viewed as a part of the whole process even in the case that an individual particle is not absorbed but goes through. Indeed, that an individual particle goes through is only a result of the state reduction, which is a change from the linear superposition of the transition and the absorption states.

5.3 The structure of meters

Here, we extend some ideas of Section 5.2 on Stern-Gerlach apparatus to all meters with the aim to improve the understanding of registrations. Most theoretical descriptions of meters that can be found in the literature are strongly idealised (see, e.g., [9, 80]): the meter is a not further specified quantum system with a “pointer” observable. We are going to give a more elaborated picture and distinguish between *fields*, *screens*, *ancillas* and *detectors* as basic structural elements of meters.

Screens have been dealt with in Section 5.2. It is also more or less clear what are fields: for example, in the Stern-Gerlach experiment, the beam is split by an inhomogeneous magnetic field. In some optical experiments, various crystals are used that make possible the split of different polarisations or the split of a beam into two mutually entangled beams such as by the down-conversion process in a crystal of KNbO_3 [52]. The corresponding crystals can also be considered as fields. In any case, the crystals and fields are macroscopic systems the (classical) state of which is not changed by the interaction with the registered system.

In many modern experiments, in particular in non-demolition and weak measurements, but not only in these, the following idea is employed. The registered system S interacts first with an auxiliary quantum system A that is prepared in a suitable state. After S and A become entangled, A is subject to further registration and, in this way, some information on S is revealed. Subsequently, further measurements on S can but need not be made. The state of S is influenced by the registration of A just because of its entanglement with A . Such auxiliary system A is usually

called *ancilla* (see, e.g., [58], p. 282).

Finally, important parts of meters are *detectors*. Indeed, even a registration of an ancilla needs a detector. It seems that any registration on microscopic systems has to use detectors in order to make features of microscopic systems visible to humans. Detector is a large system that changes its (classical) state during the interaction with the registered system. “Large” need not be macroscopic but the involved number of particles ought to be at least about 10^{10} . For example, the photo-emulsion grain or nanowire single photon detector (see, e.g., [54]) are large in this sense. A criterion for a physical object to be large is that the object has a classical model that gives a good approximation to some aspect of its behaviour. For example, the object has well-defined thermodynamic states.

For example, in the so-called cryogenic detectors [74], S interacts, e.g., with superheated superconducting granules by scattering off a nucleus in a granule. The resulting phonons induce the phase transition from the superconducting to the normally conducting phase. The detector can contain very many granules (typically 10^9) in order to enhance the probability of such scattering if the interaction between S (a weakly interacting massive particle, neutrino) and the nuclei is very weak. Then, there is a solenoid around the vessel with the granules creating a strong magnetic field. The phase transition of only one granule leads to a change in magnetic current through the solenoid giving a perceptible electronic signal.

Modern detectors are constructed so that their signal is electronic. For example, to a scintillation film, a photomultiplier is attached (as in [73]). We assume that there is a signal collected immediately after the detector changes its classical state, which we call *primary* signal. Primary signal may still be amplified and filtered by other electronic apparatuses, which can transform it into the final signal of the detector. For example, the light signal of a scintillation film in the interference experiment of [73] is a primary signal. It is then transformed into an electronic signal by a photocathode and the resulting electronic signal is further amplified by a photomultiplier.

A detector contains *active volume* \mathcal{D} and *signal collector* \mathcal{C} in thermodynamic state of metastable equilibrium. The term “thermodynamic equilibrium” is correct even in the cases in which mechanical or electrodynamic forces play a relevant role (such as that of Geiger-Muller counter). Notice that the active volume is a physical system, not just a volume of space. For example, the photo-emulsion or the set of the superconducting granules are active volumes. Interaction of the detected systems with \mathcal{D} triggers a relaxation process leading to a change of the classical state of the detector—the *detector signal*. For some theory of detectors, see, e.g., [49, 74].

What is the difference between ours and the standard ideas on detectors? The standard ideas are, e.g., stated in (Ref. [58] p. 17) with the help of the Stern-Gerlach

example:

The microscopic object under investigation is the magnetic moment μ of an atom.... The macroscopic degree of freedom to which it is coupled in this model is the centre of mass position \mathbf{r} ... I call this degree of freedom *macroscopic* because different final values of \mathbf{r} can be directly distinguished by macroscopic means, such as the detector... From here on, the situation is simple and unambiguous, because we have entered the macroscopic world: The type of detectors and the detail of their functioning are deemed irrelevant.

The root of such notion of detectors may be found among some ideas of the grounding fathers of quantum mechanics. For example, Ref. [56], p. 64, describes a measurement of energy eigenvalues with the help of scattering similar to Stern-Gerlach experiment, and Pauli explicitly states:

We can consider the centre of mass as a 'special' measuring apparatus...

In these statements, no distinction is made between ancillas and detectors: indeed, the centre-of-mass position above can be considered as an ancilla. However, such a distinction can be made and it ought to be made because it improves our understanding of registrations. Thus, to improve the understanding, we have slightly modified the current notions of detector and ancilla. Our detectors are more specific than what is often assumed.

The foregoing analysis motivates the following trial hypothesis.

Trial Hypothesis 5.1 *Any meter for microsystems must contain at least one detector and every reading of the meter can be identified with a primary signal from a detector. The state reduction required by realism and observational evidence on measurements takes place in detectors and screens.*

A similar hypothesis has been first formulated in [27]. TH 5.1 makes the reading of meters less mysterious. Moreover, TH 2.1 allows us to distinguish between the quantum and the classical models of a detector and to consider the classical model as a meter for the quantum model, so that von Neumann's chain begins and ends at the detector.

5.4 Two hypotheses on state reduction

Here, we study the form of state reduction and the objective circumstances with which it is connected.

Trial Hypothesis 5.2 *Let \mathcal{O} be an object (such as a detector) with classical model O_c and quantum model O_q . Let the standard unitary evolution describing some process in which O_q takes part results in an end state of the form:*

$$\bar{T}_f = \sum_{k=1}^n P_k \bar{T}_k + \bar{T}_{end0} , \quad (5.44)$$

where \bar{T}_k are states of O_q such that each is associated with a classical state of O_c and these classical states are different for different k 's. The coefficients satisfy $P_k > 0$ for $k = 1, \dots, n$ and $\sum_k P_k = 1$. \bar{T}_{end0} is a s.a. operator with trace 0. Then, the standard unitary evolution must be corrected so that \bar{T}_f is replaced by

$$\bar{T}_{end} = \sum_{k=1}^n P_k \bar{T}_k , \quad (5.45)$$

the proper mixture of states \bar{T}_k .

TH 5.2 is applicable to those unitary evolutions that have an end state of the form (5.44). However, classical objects may have some properties that make such a form to be a general case. For example, it may be impossible for a quantum model of a classical object to be in a convex combination of states, one of which is associated with a classical state and the other not having classical properties or in a state equal to two different convex compositions so that the two sets of classical states defined by the two compositions are different from each other. This seems to follow from the classical realism described at the beginning of Chapter 1.

To illustrate the difference to an ordinary convex decomposition, let us consider an arbitrary normalised state vector Φ of some quantum system. Such a state can be decomposed into two orthonormal vectors in an infinite number of different ways, for example,

$$\Phi = c_1 \Phi_1 + c_2 \Phi_2 = d_1 \Psi_1 + d_2 \Psi_2 .$$

Then

$$|\Phi\rangle\langle\Phi| = |c_1|^2 |\Phi_1\rangle\langle\Phi_1| + |c_2|^2 |\Phi_2\rangle\langle\Phi_2| + c_1 c_2^* |\Phi_1\rangle\langle\Phi_2| + c_2 c_1^* |\Phi_2\rangle\langle\Phi_1|$$

and

$$|\Phi\rangle\langle\Phi| = |d_1|^2 |\Psi_1\rangle\langle\Psi_1| + |d_2|^2 |\Psi_2\rangle\langle\Psi_2| + d_1 d_2^* |\Psi_1\rangle\langle\Psi_2| + d_2 d_1^* |\Psi_2\rangle\langle\Psi_1|$$

are two different decompositions of state $|\Phi\rangle\langle\Phi|$ that have the form of (5.44).

We leave the detailed questions of applicability of TH 5.2 open to future investigations in the hope that the approach that it suggests is more or less clear. In

any case, TH 5.2 defines a rule that determines the correction to unitary evolution *uniquely* in a large class of scattering and registration processes (see [28, 31]).

Both detectors and screens, where the state reductions occur, are mezzo- or macroscopic (for example, the emulsion grains can be considered as mezzoscopic), but there are processes of interaction between microscopic and macroscopic objects, the standard quantum description of which gives always a unique classical end state of the macroscopic part. For example, the scattering of neutrons by ferromagnetic crystals in which the crystal remains in the same classical state during the process of scattering. In such processes, TH 5.2 implies no state reductions. It is the structure of the final quantum state that makes the difference: for a state reduction, the standard quantum evolution had to give a convex combination of states that differ in their classical properties.

What is the cause of the change \bar{T}_f into \bar{T}_{end} ? For example, the detector that detects microsystem S achieves the signal state so that S interacts with its active volume \mathcal{D} and the state of $S+D_q$ dissipates, which leads to a loss of separation status of S . A similar process runs in a screen that absorbs S . The dissipation is necessary to accomplish the loss. The dissipation process does not have anything mysterious about it. It can be a usual thermodynamic relaxation process in a macroscopic system or a similar process of the statistical thermodynamics generalised to nano-systems (see, e.g., [39]). S might be the registered object or an ancilla of the original experiment. In all such cases, state \bar{T}_{end} originates in a process of relaxation triggered by S in D_q or B_q and accompanied by the loss of separation status of S . This motivates the following hypothesis:

Trial Hypothesis 5.3 *The cause of the state reduction postulated by TH 5.2 is an uncontrollable disturbance due to a loss of separation status.*

The loss of separation status is an objective process and the significance of TH 5.3 is that it formulates an objective condition for the applicability of an alternative kind of dynamics.

Actually, the assumption that a measuring process disturbs the measured system in an uncontrollable way and that this is the cause of the state reduction is not new (see, e.g., [53], Section 4.3.1). What we add to it is just the role of separation-status loss.

The three Trial Hypotheses 5.1, 5.2 and 5.3 form a basis of our theory of state reduction. They generalise some empirical experience, are rather specific and, therefore, testable. That is, they cannot be disproved by purely logical argument but rather by an experimental counterexample. For the same reason, they also show a specific direction in which experiments ought to be proposed and analysed: if there is a state reduction, does then a loss of separation status take part in the process? What system loses its status? How the loss of the status can lead to state reduction?

In fact, our theory remains rather vague with respect to the last question in that it suggests no detailed model of the way from a separation status change to a state reduction. Such a model would require some new physics and we believe that hints of what this new physics could be will come from attempts to answer the above questions by suitable experiments.

The new notions and hypotheses are studied on some examples and models [28], where the old definition of separation status is used. A reformulation of these examples based on the new definition seems to be more or less straightforward.

Chapter 6

Summary of RCU interpretation

The project described by this paper is to reformulate a popular version of the Copenhagen interpretation in more rigorous terms and augment it with some new ideas that make it more self consistent and comprehensible. The result is the so-called RCU interpretation. The starting point was the Minimum Interpretation, i.e., the notion that quantum mechanics was a set of rules to calculate probabilities of registration values. To this, some further assumptions were added, called Trial Hypotheses (TH). This was done step by step so that the motivation and impact of each TH could be explained. In this way, however, the whole resulting picture remained rather elusive. The purpose of this chapter is to give a short survey of all changes done.

Our first TH is Completeness Hypothesis

There are no unknown causes beyond the probabilities given by the Born rule.

It might seem more cautious if one would leave this question open but then some main principles of the standard quantum mechanics had to be changed. The Completeness Hypothesis has important consequences for interpretation of states.

The next aim is to provide a tool against the positivist and instrumentalist standpoint to which quantum physicists are motivated by Minimum Interpretation. To accomplish this task, we lean strongly on the realist version of the model approach to the philosophy of science, the Constructive Realism [20]. This leads us to distinguish between a real object and its models, in particular physical objects and their classical and quantum models.

Our definition of quantum (real) object, TH 1.1, reads:

A quantum object is defined by a preparations. The objects are thus distinguished from each other by the properties that are determined by their preparations. These include the structural properties describing a system type, the prepared state and properties that are uniquely defined

by the state. Objects of non-relativistic quantum mechanics can be classified into electrons, neutrons, nuclei, atoms, molecules and their composites.

This represents our requirement that real objects must have a sufficient number of objective properties and our basic assumption that such properties of quantum systems are those defined uniquely preparations. TH 1.1 strongly influenced almost all further development. In particular, it motivates the interpretation of observable values in TH 1.2 (Outcomes Created by Registration):

The outcome of an individual registration performed on a quantum object S in state T is in general only created during the registration. It is an objective property of the whole registration process, not of the registered system.

TH 1.2 also helps explain some quantum paradoxes, such as the wave-particle dualism or existence of a special quantum “logic”, which is no logic but an algebra of projection operators so that working with this algebra requires the ordinary mathematical logic.

TH 1.1 is further specified by TH 1.5:

Objective properties of a quantum system S can be divided into three classes: 1. structural properties of S , 2. a state of S and the properties determined uniquely by the state (such as expectation values of a fixed observable), 3. the properties of the state of a system that has been prepared so that it contains S as a subsystem if such properties concern S but are not determined by the state of S itself (such as the way S is entangled with other systems).

Structural properties are those that are common to all systems of the same type, such as mass and charge. The TH gives a list of objective properties. It also includes the correlations between pairs of observables, each belonging to one of two entangled systems.

We have derived some important consequences of TH 1.1: the existence of proper mixtures and the fundamental distinction between proper mixtures and all other quantum states. We have motivated the notion that the popular distinction between pure states and mixtures is rather misleading. Instead, leaning on the Completeness Hypothesis, we introduce the distinction between ontic and epistemic states as the only physically meaningful one by TH 1.4:

Extremal states and improper mixtures are ontic, proper mixtures are epistemic, quantum states.

Hence, the quantum states that are not proper mixtures give a maximum information of quantum systems and have by themselves, no statistical character.

The universality of quantum mechanics, TH 1.3, reads:

Every object has a quantum model that accounts for all its known physical properties.

It also postulates that quantum mechanics is applicable to classical systems. Chapters 2 and 3 start a new approach to the problem of construction of quantum models describing classical systems. The basis is TH 2.1:

Let S_c be a classical model and S_q a quantum model of object S . Then S_c can be considered as an essential part of the preparation device and, simultaneously, as an essential part of a meter, for the quantum model S_q . The meter in question registers values distinguishing the quantum states of S_q that are associated with different classical states of S_c .

It relates the classical model to the preparation and registration apparatus for the quantum model. TH 2.1 also illustrates the practical value and advantage of the model approach.

The relation between quantum and classical properties is postulated by TH 2.2:

Let a real object \mathcal{S} has a classical model S_c and a quantum model S_q . Then all properties of S_c are selected properties of some high-entropy states of S_q .

This is motivated by thermodynamics, where it is well-known to be valid, and contradicts sharply the common prejudice that quantum coherent states carry classical properties. For the special case of Newtonian mechanics, the idea that we call Exner-Born Conjecture is stressed:

States of Newtonian systems that are described by sharp points of the phase space do not exist. Newtonian models that can approach the reality better are non-trivial probability distribution function on the phase space.

The conjecture opens the way to a unified theory for both thermostatic and mechanical properties. The technical tool is the ME-packet notion, the state maximizing entropy for given averages and variances, and its significance is expressed by the ME-Packet Conjecture:

For most mechanical objects \mathcal{S} , all measurable predictions of Newtonian mechanics can be obtained from a classical model S_c , that is an ME packet defined by Theorem 2.1.

It allows to choose specific high-entropy states of Newtonian systems—ME packets—as those that are to be approximated by quantum models.

The classical and quantum mathematics of maximum entropy packets has been developed. Its main result (Theorem 3.2) is that the trajectories of classical and quantum ME packets match each other the better, the higher their entropy is. That is, the fuzziness improves the match between classical and quantum theory. This confirms the feeling that quantum mechanics is more accurate and finer than Newtonian mechanics. Newtonian mechanics also becomes conspicuously similar to phenomenological thermodynamics. The result is proved for all polynomial potential functions. We hope to be able to prove this statement for non-polynomial potentials such as Coulomb potentials associated with Kepler orbits. Also, the idea ought yet to be generalised to classical electro- and magnetostatic properties, as well as to the relativistic classical electrodynamics.

The theory also suggests promising ideas of how the well-known conceptual problems associated with classical properties (explained at the start of Chapter 2) could be solved.

Next, we turn to measurement theory. Chapter 4 studies the well-known (c.f. [58]) but generally ignored disturbance of registrations of system S by environmental particles of the same type. We have shown that it can only be avoided if the registration is made by incomplete meters. The incomplete apparatus must give probability zero to all values that are registered on states of the environment. Moreover, the measured system must be prepared in a state that lifts it from the sea of identical particles: it must have the so-called separation status, Definition 4.10. Then, the environmental particles that are indistinguishable from the measured system can be ignored in the practice of registrations and in their theoretical treatment, as it is usually done.

This leads to a new notion of preparation process, TH 4.1:

Any preparation of a single microsystem must yield a state having a separation status.

This correction concerns all previous Trial Hypotheses that use the notion of preparation. For instance, it refines Trial Hypothesis 1.1 about objectivity of structural properties and prepared states concerning the word “prepared” in it.

The definition of separation status proposed in Chapter 4 is different from that of [27, 28] so that some problems of the old definition are removed. Moreover, the new notion of separation status is more general and simpler to use than the old one.

The relation between a quantum observable and its registration apparatus becomes then more complicated than is usually assumed. The observables themselves are defined as in the standard quantum theory (as self-adjoint operators). In this way, the simplicity and elegance of the standard mathematics (such as C^* -algebras)

of quantum observables is preserved. However, the role of meters was described by Trial Hypothesis 4.2:

For any observable O of a system S , there is a meter that can register O . There is no meter that can register O on any state.

This means that each observable represents a whole set of apparatuses, each registering only the part of it that is associated with a subset of states. We do not want to deny that some measurements are better described by positive-operator valued measures (POVM) than by self-adjoint operators but just do not call POVM “observables”. The relation between observables and apparatuses described by TH 4.2 is also valid for that of POVM and apparatuses.

The new theory is logically consistent with the Minimum Interpretation of exchange symmetry, agrees with the praxis of real measurements and our understanding of registration apparatuses as well as that of preparation processes is improved.

A reformulation of the quantum theory of measurement theory is given in Chapter 5. Section 5.4 gives a narrower definition to the notion of detector (already proposed in [27]) than what is usually assumed. The basic assumption is TH 5.1:

Any meter for microsystems must contain at least one detector and every reading of the meter can be identified with a primary signal from a detector. The state reduction required by realism and observational evidence on measurements takes place in detectors and screens.

The form of state reduction is uniquely specified by TH 5.2:

Let \mathcal{O} be an object (such as a detector) with classical model O_c and quantum model O_q . Let the standard unitary evolution describing some process in which O_q takes part results in an end state of the form:

$$\bar{T}_f = \sum_{k=1}^n P_k \bar{T}_k + \bar{T}_{\text{end}0} ,$$

where \bar{T}_k are states of O_q such that each is associated with a classical state of O_c and these classical states are different for different k 's. The coefficients satisfy $P_k > 0$ for $k = 1, \dots, n$ and $\sum_k P_k = 1$. $\bar{T}_{\text{end}0}$ is a s.a. operator with trace 0. Then, the standard unitary evolution must be corrected so that \bar{T}_f is replaced by

$$\bar{T}_{\text{end}} = \sum_{k=1}^n P_k \bar{T}_k ,$$

the proper mixture of states \bar{T}_k .

There are two innovations in TH 5.2: first, it specifies the form of state reductions uniquely and second, it postulates state reduction as a transformation between high-entropy states of large systems. Thus, it needs more sophisticated mathematical methods, which are introduced in Sections 5.2 and 5.3. It is also necessary to stress that our theory leads to some doubts on the standard theory of quantum measurement as it is explained e.g. in [80]. The basic notion of the standard theory is that of “state transformer”: a transformation between the initial state of the measured system S and the state of S after the registration by detector M . We have shown that the registered system is lost after the registration, that is it cannot be identified with any well-defined subsystem of the composed system $S + M$.

A suggestion that the cause of state reduction may be associated with a loss of separation status is TH 5.3:

The cause of the state reduction postulated by TH 5.2 is an uncontrollable disturbance due to a loss of separation status.

It is a more precise form of the original suggestion in [27]. It promotes the state reduction to a physical process, specifies the objective conditions of its occurrence and shows where and when it takes place. However, it does not give a detailed account of the process.

Many applications of our measurement theory are described in [28]. It is also shown there that the theory has measurable consequences and a direction of possible further development is suggested.

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